A topos for étale-finite Heyting algebras

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Joint work with Rodrigo N. Almeida and Igor Arrieta

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Recording of the talk available at this link:
`https://youtu.be/_oA9ZzpNGlg`

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Consider sentences modulo provable equivalence from T; order them by provability.

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Example. How to obtain $\mathbf{2}$?

Take any consistent complete theory, such as:

- the theory of a one-element set (empty language);
- dense linear orders without endpoints.

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I.e., for which posets X is there an *elementary topos* $\mathcal{E}$ such that X is isomorphic to Subterminals($\mathcal{E}$), the poset of subobjects of the terminal object?

Subterminals($\mathcal{E}$) is called the *lattice of truth values* of $\mathcal{E}$.

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Terminology

A Heyting algebra H is *topos-admissible* if $H \cong$ Subterminals($\mathcal{E}$) for some elementary topos $\mathcal{E}$.

Is every Heyting algebra topos-admissible?

Question (... Freyd $\xrightarrow{1980}$ Pitts $\xrightarrow{1992}$ world)

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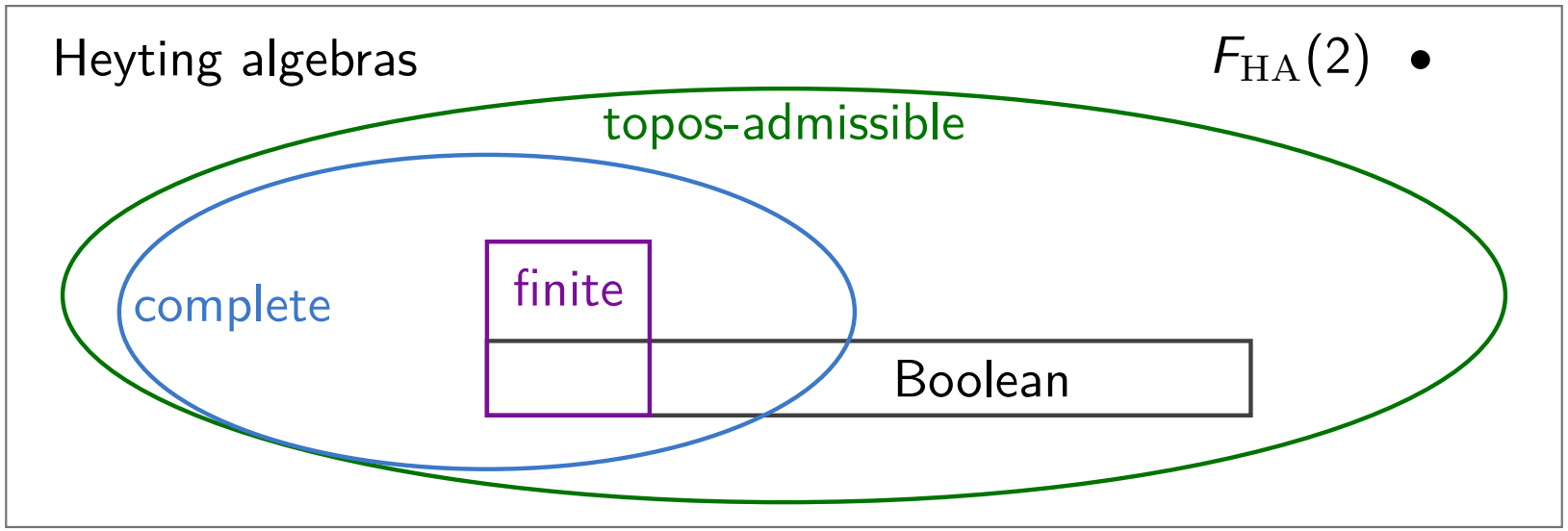
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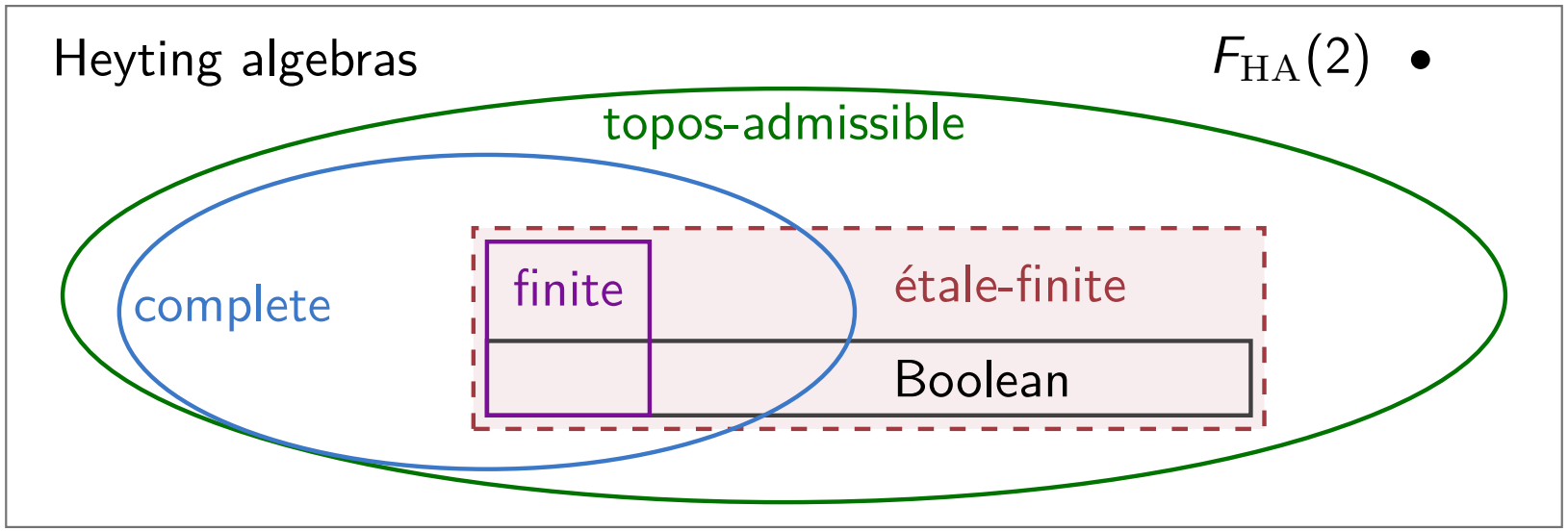
Necessary conditions for topos-admissibility? Sufficient conditions?

Some classes of Heyting algebras



Complete and Boolean: known positive cases.

Some classes of Heyting algebras



Complete and Boolean: known positive cases.

Étale-finite: the class considered in this talk.

Complete Heyting algebras

Let H be a complete Heyting algebra, viewed as a *locale*.

The category $\mathcal{E}$ of sheaves on this locale satisfies

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For example, if $H = \mathcal{O}(X)$ for a topological space X,

$$\mathrm{Subterminals}(\mathrm{Sh}(X)) \cong \mathcal{O}(X).$$

Sheaves as étale spaces

A sheaf on X is equivalently a *local homeomorphism*

$$\begin{array}{c} Y \\ \downarrow \\ X \end{array}$$

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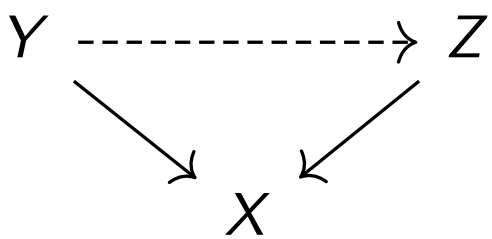
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Morphisms:



Subterminal objects and open subsets

Monomorphisms are exactly the *injective maps*.

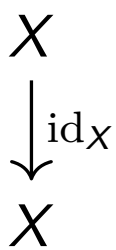
Terminal object

$$\begin{array}{c} X \\ \downarrow{\scriptstyle \mathrm{id}_X} \\ X \end{array}$$

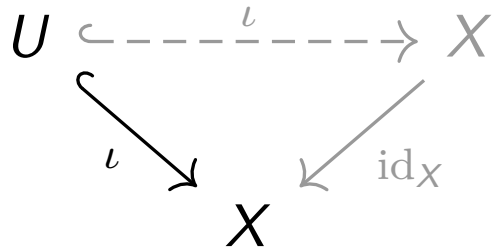
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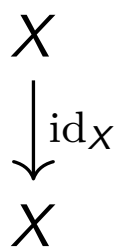
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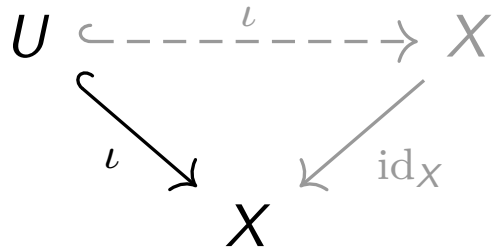
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Terminal object



Subterminal object



Local homeomorphisms are *open maps*.

Thus subterminal objects correspond to *open subsets* of X.

$$\mathsf{Subterminals}(\mathsf{Sh}(X)) \cong \mathcal{O}(X).$$

Boolean algebras: Freyd's construction

Let B be a Boolean algebra and X its Stone dual. Usual sheaves give $\mathcal{O}(X)$ as lattice of truth values. How to obtain $B \cong \mathrm{Clop}(X)$?

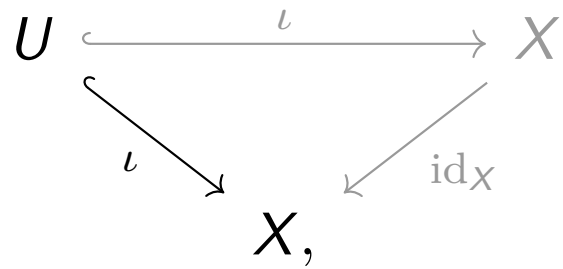
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$$\cancel{\mathrm{Top}_{\mathrm{LH}}/X} \qquad \mathrm{Stone}_{\mathrm{LH}}/X.$$

Objects of $\mathrm{Stone}_{\mathrm{LH}}/X$: local homeomorphisms $Y \to X$, with **also** Y a *Stone space*.

This is an elementary topos ($\approx$ Freyd). Given a subterminal object



U must be open (since local homeomorphisms are open) and closed (since U is a Stone space).

Beyond Boolean algebras

We generalise this trick to the *largest class of Heyting algebras* for which it works.

Opens and compact opens

Let L be a bounded distributive lattice, and X its dual *spectral space*.

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Idea: take spectral local homeomorphisms $Y \to X$ with also Y a *spectral space*.

$\mathrm{Spec}_{\mathrm{LH}}$: spectral spaces and *local homeomorphisms*. We consider

$$\mathrm{Spec}_{\mathrm{LH}}/X.$$

The main result

Every *étale-finite Heyting algebra* is *topos-admissible*.

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We prove something more:

Theorem (A.–Almeida–Arrieta)

Let L be a *bounded distributive lattice* and X its *dual spectral space*.

$$\mathsf{Subterminals}(\mathsf{Spec}_{\mathrm{LH}}/X) \cong L.$$

The slice category $\mathsf{Spec}_{\mathrm{LH}}/X$ is an *elementary topos* if and only if L is an *étale-finite Heyting algebra*.

Étale-finite Heyting algebras

A generalisation of the *law of excluded middle*:

Definition (Essentially due to Evgeny Kuznetsov)

A Heyting algebra H is *étale-finite* if there is a finite subset $S \subseteq H$ such that, for every $x \in H$,

$$\bigvee_{s \in S} (x \leftrightarrow s) = 1.$$

Here $x \leftrightarrow s = (x \to s) \wedge (s \to x)$.

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For $S = \{0, 1\}$ this says

$$\text{(Excluded middle)} \quad \neg x \vee x = 1.$$

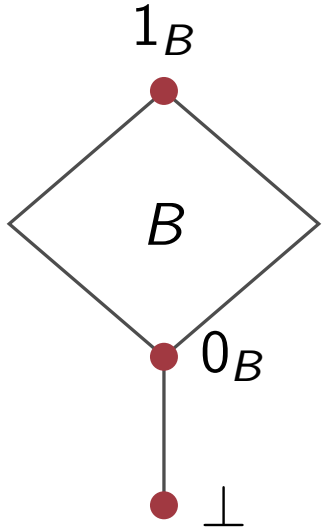
Examples

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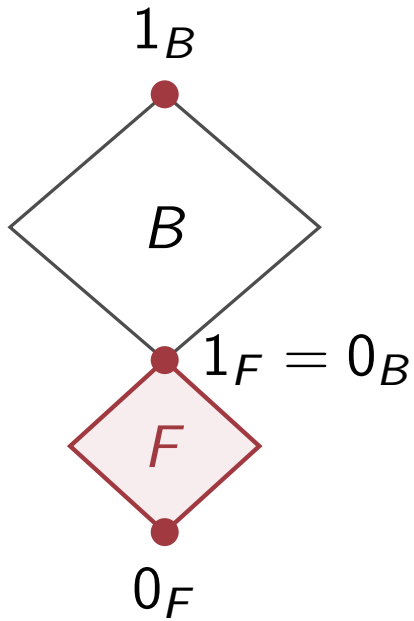
Also: a Boolean algebra with a *new bottom element*.



$S = \{\bot, 0_B, 1_B\}$.

Examples

More generally, place a *finite Heyting algebra* F below a Boolean algebra B.



$S = F \cup \{1_B\}$.

A dual description

Let X be the dual Priestley space of a Heyting algebra H.

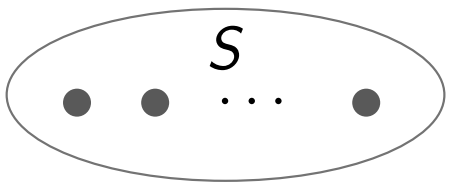
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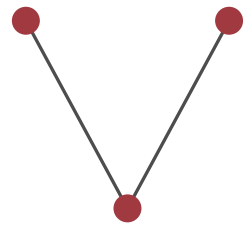
Boolean algebras



Stone space, equality order.

$$\uparrow x = \{x\}.$$

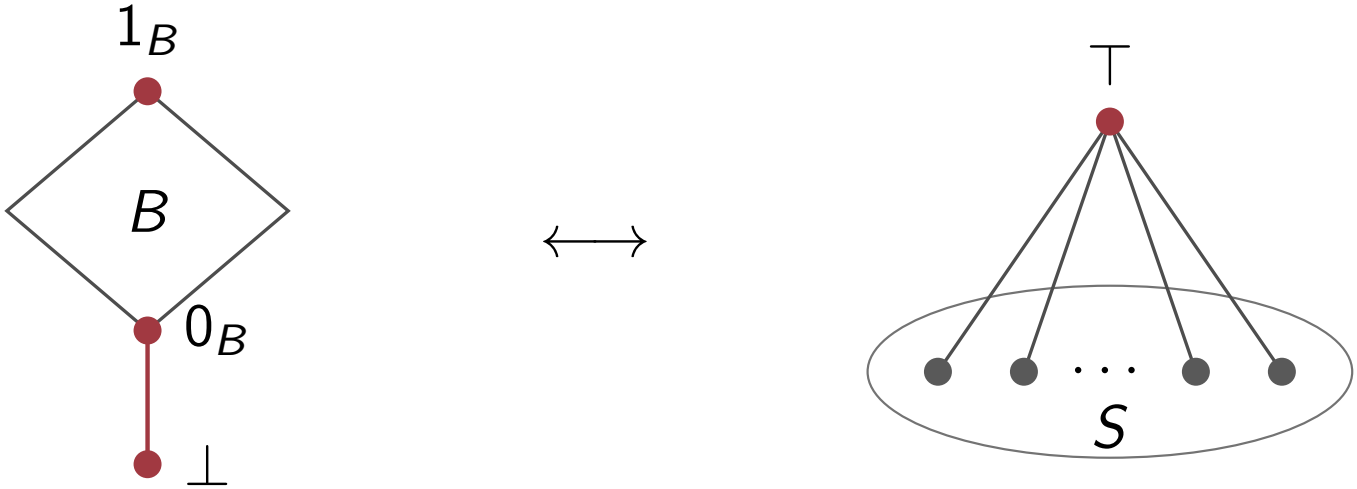
Finite Heyting algebras



Finite poset,
discrete topology.

A new bottom, dually

Adding a new bottom to B corresponds to adding an *isolated greatest point* to its Stone dual S.



A topos for étale-finite Heyting algebras

Recall: an elementary topos is a category with *finite limits* and *power objects*.

Let L be a bounded distributive lattice, X its dual spectral space, and $\mathcal{E} = \mathrm{Spec}_{\mathrm{LH}}/X$.

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Easy

- Subterminals$(\mathcal{E}) \cong L$;
- $\mathcal{E}$ has *finite limits*.

Difficult

$\mathcal{E}$ has *power objects* if (and only if) L is an *étale-finite Heyting algebra*.

Obstruction beyond étale-finiteness

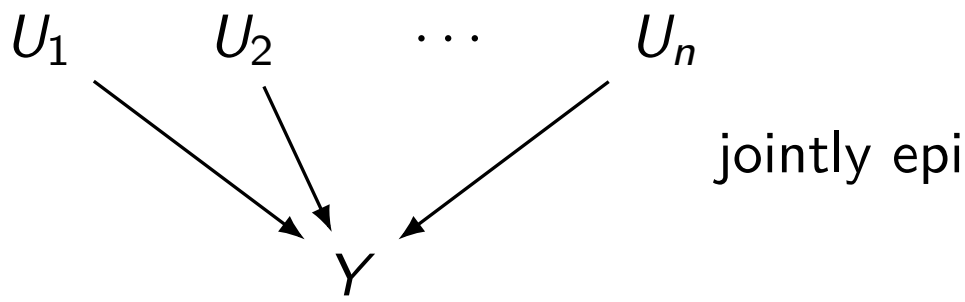
Let L be a bounded distributive lattice and X its dual spectral space.

$$\mathrm{Spec}_{\mathrm{LH}}/X \text{ is an } \textit{elementary topos}$$

$$\Updownarrow$$

L is an *étale-finite Heyting algebra*.

In our category, every object Y has a *finite cover by subterminals*:

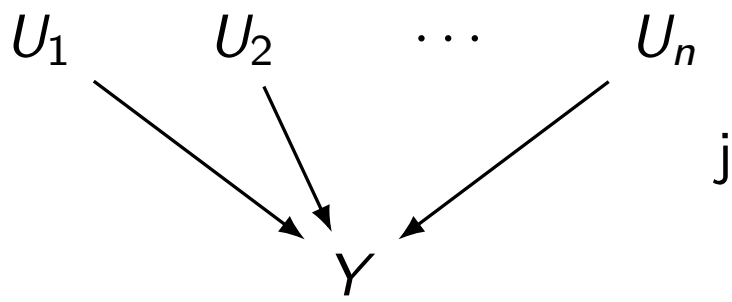


jointly epi

$U_i \hookrightarrow 1.$

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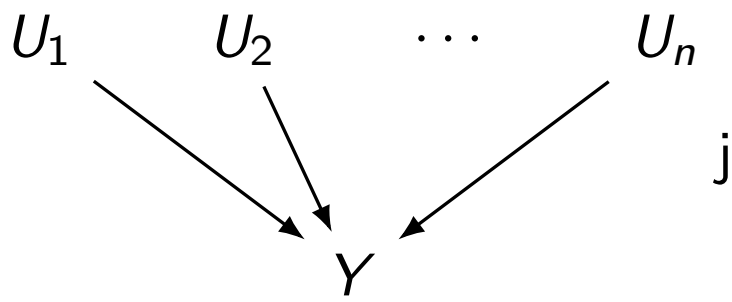
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If an elementary topos is finitely propositional, then its lattice of truth values is étale-finite.

Proof: Apply the covering property to $Y = \Omega$... □

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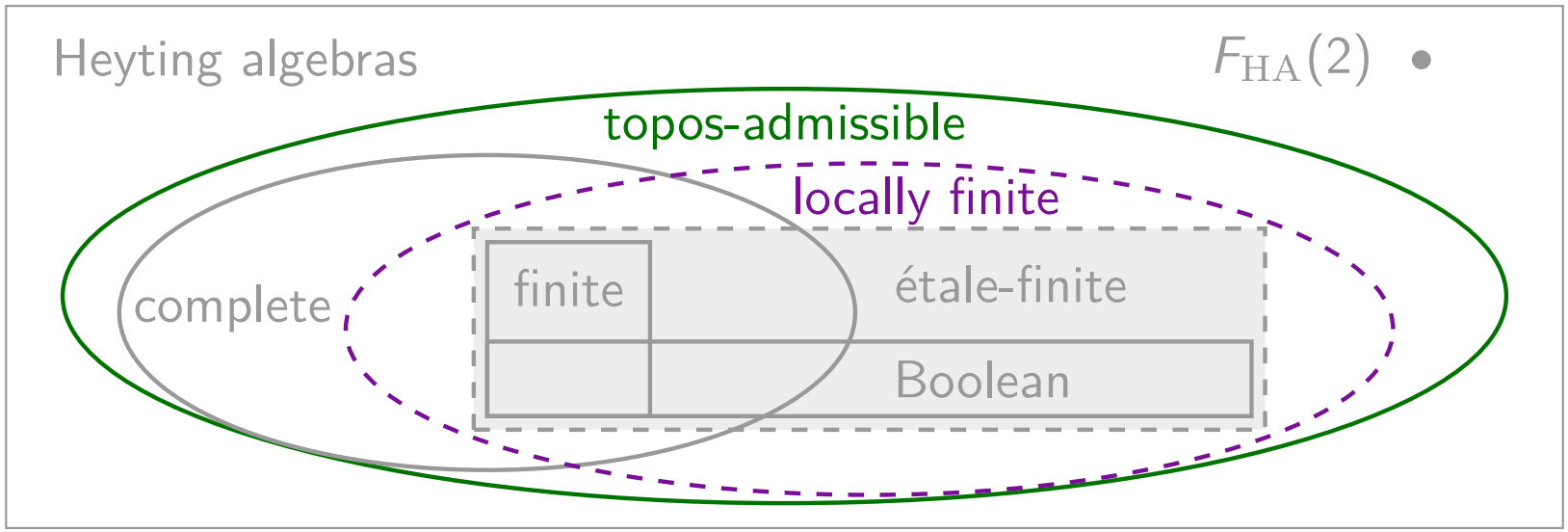
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Proof: Apply the covering property to $Y = \Omega$... □

So this *compact étale-space method* reaches exactly the étale-finite Heyting algebras.

Work in progress

Locally finite Heyting algebras: work in progress



Work in progress

Every **locally finite** Heyting algebra is topos-admissible.

For example, all bounded chains.

The construction is more involved, also in the étale-finite case.

Summing it up

Main results

- Every étale-finite Heyting algebra is topos-admissible, via a slice $\mathrm{Spec}_{\mathrm{LH}}/X$ of the category of spectral spaces and local homeomorphisms.
- Methods based on compact étale spaces can only work for étale-finite Heyting algebras.

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Abbadini, Almeida, Arrieta,

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Thank you!