Is the category of locally finite MV-algebras equivalent to an equational class?

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MV-algebras

An *MV-algebra* is an algebra in $\mathbb{HSP}(\langle [0,1]; \oplus, \neg, 0\rangle)$, where

$$\begin{aligned} x \oplus y &:= \min\{x+y, 1\}, \\ \neg x &:= 1 - x, \\ 0 &:= \text{the element } 0. \end{aligned}$$

(MV-algebras are the unit intervals of Abelian lattice-ordered groups with strong order unit.)

Locally finite algebras

Definition

An algebra is called *locally finite* if every finitely generated subalgebra is finite.

Locally finite MV-algebras? Let us first describe the simplest locally finite MV-algebras: those which can be embedded in $[0,1]$.

The MV-algebra $[0,1]$ is not locally finite: for any irrational $x \in [0,1]$, the subalgebra of $[0,1]$ generated by x is infinite.

- For $n \in \mathbb{N}_{>0}$, $\{\frac{0}{n}, \frac{1}{n}, \dots, \frac{n-1}{n}, \frac{n}{n}\}$ is locally finite.
- $[0,1] \cap \mathbb{Q}$ is locally finite.

Locally finite subalgebras of $[0,1]$ = subalgebras of $[0,1] \cap \mathbb{Q}$.

- $\left\{\frac{i}{2^k} \mid k \in \mathbb{N}, i \in \{0, \dots, 2^k\}\right\}$ is locally finite.

Subalgebras of $[0,1] \cap \mathbb{Q}$ can be encoded by *supernatural numbers.*

Encoding of subalgebras of $[0,1] \cap \mathbb{Q}$

A finite subalgebra $\{\frac{0}{n}, \frac{1}{n}, \dots, \frac{n-1}{n}, \frac{n}{n}\}$ of $[0,1] \cap \mathbb{Q}$ can be encoded by n.

An arbitrary subalgebra of $[0,1] \cap \mathbb{Q}$ can be encoded by a "supernatural number", which generalizes the prime factorization of a positive integer.

$\mathbb{P} :=$ set of pime numbers.

Definition

A *supernatural number* is a function $\mathbb{P} \to \mathbb{N} \cup \{\infty\}$.

(Convention: $\mathbb{N} := \{0, 1, 2, \dots\}$, $\mathbb{N}_{>0} := \{1, 2, \dots\}$.)

$\mathcal{N} :=$ set of supernatural numbers.

A quotient of $\mathcal{N}$ is used to classify torsion-free groups of rank 1.

Injection $\mathbb{N}_{>0} \hookrightarrow \mathcal{N}$ mapping $n \in \mathbb{N}_{>0}$ to its prime factorization

$$\begin{aligned} \nu_n \colon \mathbb{P} &\longrightarrow \mathbb{N} \cup \{\infty\} \\ p &\longmapsto \max\left\{j \in \mathbb{N} \mid p^j \text{ divides } n\right\}. \end{aligned}$$

E.g.,
$$\nu_{12}(p) = \begin{cases} 2 & \text{if } p = 2 \\ 1 & \text{if } p = 3 \\ 0 & \text{otherwise.} \end{cases}$$

A supernatural number ν is said to be *finite* if there exists $n \in \mathbb{N}_{>0}$ such that $\nu = \nu_n$, i.e.

- ∞ does not belong to the range of ν,
- $\nu(p) = 0$ for all but finitely many $p \in \mathbb{P}$.

$\mathcal{N}$ is equipped with the pointwise order:

$$\nu \leq \nu' \overset{\text{def}}{\iff} \forall p \in \mathbb{P} \;\; \nu(p) \leq \nu'(p).$$

The order on $\mathcal{N}$ generalizes the divisibility order of $\mathbb{N}_{>0}$ (and corresponds to the inclusion of subalgebras of $[0, 1] \cap \mathbb{Q}$).

$\{\text{subalgebras of } [0,1] \cap \mathbb{Q}\} \longleftrightarrow \mathcal{N}$

- To a subalgebra A of $[0,1] \cap \mathbb{Q}$, we associate the supernatural number

$$\begin{aligned} \nu \colon \mathbb{P} &\longrightarrow \mathbb{N} \cup \{\infty\} \\ p &\longmapsto \sup\left\{ j \in \mathbb{N} \mid \frac{1}{p^j} \in A \right\}. \end{aligned}$$

  E.g.: to $\left\{\frac{0}{12}, \frac{1}{12}, \dots, \frac{11}{12}, \frac{12}{12}\right\}$ we associate ν_{12}.
  E.g.: to $\left\{\frac{i}{2^k} \mid k \in \mathbb{N}, i \in \{0, \dots, 2^k\}\right\}$ we associate the supernatural number ν s.t. $\nu(p) = \begin{cases} \infty & \text{if } p = 2 \\ 0 & \text{otherwise.} \end{cases}$

- To a supernatural number $\nu \colon \mathbb{P} \longrightarrow \mathbb{N} \cup \{\infty\}$ we associate the subalgebra

$$\{x \in [0,1] \cap \mathbb{Q} \mid \nu_{\mathrm{den}(x)} \leq \nu\}.$$

Representation

Theorem [Cignoli, Dubuc, Mundici, 2003]

An MV-algebra A is locally finite iff there is a set I such that A is isomorphic to a subalgebra $\iota(A)$ of the MV-algebra $([0,1]\cap\mathbb{Q})^I$ formed by functions of finite range.

Let I be a set. For every $i \in I$, let A_i be a subalgebra of the MV-algebra $[0,1]\cap\mathbb{Q}$. Set

$$A \coloneqq \{f\colon I \to [0,1]\cap\mathbb{Q} \mid (\forall i \in I\ f(i) \in A_i),\ \mathrm{Im}(f) \text{ is finite}\}.$$

Topologizing $\rightsquigarrow$ duality [Cignoli, Dubuc, Mundici, 2003].

Topology on $\mathcal{N}$: take as elements of a closed sub-basis for $\mathcal{N}$ all sets of the form

$$\{\nu \in \mathcal{N} \mid \nu(p) \leq k\}$$

for $p \in \mathbb{P}$ and $k \in \mathbb{N}$.

Definition

A *multiset* is a pair (X, ζ), where X is a Stone space, and $\zeta \colon X \to \mathcal{N}$ is a continuous map.

For $x \in X$, the supernatural number $\zeta(x)$ is called the *denominator* of x.

Given a multiset (X, ζ), we obtain a locally finite MV-algebra

$$\{f \colon X \to [0,1] \cap \mathbb{Q} \mid f \text{ is cont., } \mathrm{Im}(f) \text{ is finite}, \forall x \in X \; f(x) \in A_x\},$$

where A_x is the subalgebra of $[0,1] \cap \mathbb{Q}$ associated with $\zeta(x)$.

- $\mathsf{MV}_{\text{loc.fin.}}$:= category of locally finite MV-algebras and homomorphisms.
- MultiSet := category of multisets and denominator-decreasing continuous functions.
  A function $f\colon (X, \zeta_X) \to (Y, \zeta_Y)$ *decreases denominators* if

$$\forall x \in X \quad \zeta_Y(f(x)) \leq \zeta_X(x).$$

Theorem [Cignoli, Dubuc, Mundici, 2003]

The categories

- $\mathsf{MV}_{\text{loc.fin.}}$ of locally finite MV-algebras, and
- MultiSet of multisets

are dually equivalent.

- A homomorphic image of a locally finite algebra is locally finite.
- A subalgebra of a locally finite algebra is locally finite.
- The product of finitely many locally finite algebras is locally finite.

$\Rightarrow$ The class of locally finite MV-algebras is closed under homomorphic images, subalgebras and finite products.

- The class of locally finite MV-algebras is not closed under arbitrary products: e.g., $[0,1] \cap \mathbb{Q}$ is locally finite, but, for any infinite set X, $([0,1] \cap \mathbb{Q})^X$ is not locally finite.
- $\mathsf{MV}_{\text{loc.fin.}}$ is complete and cocomplete. In particular, it admits all products in the categorical sense.
  E.g.: for any set X, the categorical product of $|X|$ many copies of $[0,1] \cap \mathbb{Q}$ is

$$\{f \in ([0,1] \cap \mathbb{Q})^X \mid \mathrm{Im}(f) \text{ is finite}\}.$$

The class of locally finite MV-algebras is closed under homomorphic images, subalgebras and finite products; furthermore, $\mathsf{MV}_{\text{loc.fin.}}$ admits all products in the categorical sense.

Question [Mundici, 2011]

Is the category of locally finite MV-algebras equivalent to an equational class?

Answer: it depends. Finitary or infinitary algebras? Do we allow (possibly infinitely) many sorts?

Theorem (Negative result)

$\mathsf{MV}_{\text{loc.fin.}}$ is not equivalent to any finitely-sorted quasi-variety of finitary algebras (let alone a single-sorted variety of finitary algebras).

Theorem (Positive results)

1. $\mathsf{MV}_{\text{loc.fin.}}$ is equivalent to a single-sorted variety of infinitary algebras, with operations of at most countable arity.
2. $\mathsf{MV}_{\text{loc.fin.}}$ is equivalent to a countably-sorted variety of finitary algebras.

Next part of the talk: sketch of the proofs, rough description of a variety that satisfies 2.

We use characterizations of categories equivalent to (single/many-sorted) varieties of (finitary/possibly infinitary) algebras.

We test whether the duals of these characterizations hold in MultiSet.

In the positive cases, the proof can be used to obtain a description of the variety.

Warning: the forgetful functor $\mathsf{MV}_{\mathsf{loc.fin.}} \to \mathsf{Set}$ does not preserve products, and hence any possible equivalence between $\mathsf{MV}_{\mathsf{loc.fin.}}$ and an equational class is not concrete.

For our positive results, we use the following characterizations.

Theorem [Lawvere, 1963; Isbell, 1964, ...]

Let C be a locally small category.

1. C is equivalent to a single-sorted variety of (possibly infinitary) algebras iff C is cocomplete, Barr-exact and C has a regular projective regular generator.
2. C is equivalent to a many-sorted variety of finitary algebras iff C is cocomplete, Barr-exact and C has an abstractly finite regularly generating set of regular projective objects.

1. Special object: free algebra over a singleton.
2. Special set of objects: for each sort, the free algebra over an element placed in that particular sort.

We prove that MultiSet is co-(Barr-exact).

1. We exhibit a co-(regular projective regular generator) in MultiSet.
2. We exhibit a co-(abstractly finite regularly generating set of regular projective objects) in MultiSet.

Theorem (Positive results)

1. $\mathsf{MV}_{\text{loc.fin.}}$ is equivalent to a single-sorted variety of infinitary algebras, with operations of at most countable arity.
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Proposition

A set of objects $\mathcal{G}$ in MultiSet is a co-(abstractly finite regularly generating set of projective objects) if (and only if?) the following conditions hold.

1. For every object X of $\mathcal{G}$, the underlying set of X is finite, the denominator of each point of X is finite, and there exists an element of denominator ν_1.
2. There exists an object $X \in \mathcal{G}$ with two distinct elements of denominator ν_1.
3. For all $p \in \mathbb{P}$, $k \in \mathbb{N}_{>0}$, there exists $x \in X \in \mathcal{G}$ with denominator ν_{p^k}.

Notation

For $n \in \mathbb{N}_{>0}$, we define the multiset $\mathrm{D}_n = \{0, 1\}$ with $\zeta(0) = \nu_1$ and $\zeta(1) = \nu_n$.

An example of a co-(abstractly finite regularly generating set of projective objects) in MultiSet is the set $\{\mathrm{D}_n \mid n \in \mathbb{N}_{>0}\}$.

Choosing this set leads to the countably-sorted variety of finitary algebras in the following slides.

We describe a countably-sorted finitary clone $\mathbf{A}$ such that

$$\mathbb{SP}(\mathbf{A}) = \mathbb{HSP}(\mathbf{A}) \overset{\mathrm{op}}{\cong} \mathsf{MultiSet}.$$

Set of sorts = $\mathbb{N}_{>0}$.
In each sort, the value of $\mathbf{A}$ is $\{0,1\}$. For $s_1, \dots, s_n, t \in \mathbb{N}_{>0}$, the operations of arity $(s_1 \dots s_n, t)$ are the functions

$$f \colon \underbrace{\{0,1\}}_{\text{sort } s_1} \times \cdots \times \underbrace{\{0,1\}}_{\text{sort } s_n} \longrightarrow \underbrace{\{0,1\}}_{\text{sort } t}$$

such that, for every $(x_1, \dots, x_n) \in \{0,1\} \times \cdots \times \{0,1\}$,

$$t^{f(x_1,\dots,x_n)} \text{ divides } \operatorname{lcm}\{s_i^{x_i} \mid i \in \{1,\dots,n\}\},$$

i.e.,

$$\text{either } f(x_1, \dots, x_n) = 0$$
$$\text{or } t \text{ divides } \operatorname{lcm}\{s_i \mid i \in \{1,\dots,n\}, x_i = 1\}.$$

- Case $s_1 = \cdots = s_n = t = 1$.
  The operations of arity $(1 \dots 1, 1)$ are all the functions

$$f\colon \underbrace{\{0,1\}}_{\text{sort } 1} \times \cdots \times \underbrace{\{0,1\}}_{\text{sort } 1} \longrightarrow \underbrace{\{0,1\}}_{\text{sort } 1}$$

  i.e. all Boolean operations.
  Thus, in sort 1 we have a Boolean ring (operations generated by $0, +, \cdot, 1$).

- Case $s_1 = \cdots = s_n = t \neq 1$.
  The operations of arity $(t \dots t, t)$ are all the functions

$$f\colon \underbrace{\{0,1\}}_{\text{sort } t} \times \cdots \times \underbrace{\{0,1\}}_{\text{sort } t} \longrightarrow \underbrace{\{0,1\}}_{\text{sort } t}$$

  such that $f(0, \dots, 0) = 0$.
  Thus, in sort $t \neq 1$ we have a Boolean rng (operations generated by $0, +, \cdot$).

- The identity function

$$\mathrm{id}\colon \underbrace{\{0,1\}}_{\text{sort } s} \longrightarrow \underbrace{\{0,1\}}_{\text{sort } t}$$

  is an operation of $\mathbf{A}$ iff t divides s.
- For all $s_1, s_2 \in \mathbb{N}_{>0}$, we have an operation

$$\begin{aligned} \cdot\colon \underbrace{\{0,1\}}_{\text{sort } s_1} \times \underbrace{\{0,1\}}_{\text{sort } s_2} &\longrightarrow \underbrace{\{0,1\}}_{\text{sort } \mathrm{lcm}\{s_1,s_2\}} \\ (x,y) &\longmapsto x \cdot y. \end{aligned}$$

(The operations described above and in the previous slide generate the clone $\mathbf{A}$?)

Then,

$$\mathbb{SP}(\mathbf{A}) = \mathbb{HSP}(\mathbf{A}) \overset{\text{op}}{\cong} \mathsf{MultiSet} \overset{\text{op}}{\cong} \mathsf{MV}_{\text{loc.fin.}}$$

$\mathbb{SP}(\mathbf{A})$	MultiSet	$\mathsf{MV}_{\text{loc.fin.}}$
$\mathbf{A}$	$\{*\}$ with denominator $p \mapsto \infty$	$[0,1] \cap \mathbb{Q}$
subalg. of $\mathbf{A}$	$\{*\}$ with some denominator	subalg. of $[0,1] \cap \mathbb{Q}$

E.g.: the subalgebra $\{0,1\} \subseteq [0,1] \cap \mathbb{Q}$ corresponds to the subalgebra $\mathbf{B}$ of $\mathbf{A}$ whose value in sort n is

$$\begin{cases} \{0,1\} & \text{if } n = 1 \\ \{0\} & \text{otherwise.} \end{cases}$$

E.g.: the subalgebra $\left\{ \frac{i}{2^k} \mid k \in \mathbb{N}, i \in \{0, \dots, 2^k\} \right\} \subseteq [0,1] \cap \mathbb{Q}$ corresponds to the subalgebra $\mathbf{D}$ of $\mathbf{A}$ whose value in sort n is

$$\begin{cases} \{0,1\} & \text{if } \exists k \in \mathbb{N} \text{ s.t. } n = 2^k, \\ \{0\} & \text{otherwise.} \end{cases}$$

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Thank you for your attention!