The opposite of the category of compact ordered spaces as an infinitary variety

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Compact ordered spaces

Compact ordered spaces: introduced by [Nachbin, 1948] as an ordered version of compact Hausdorff spaces.

Definition [Nachbin, 1948]

A *compact ordered space* is a compact space X with a partial order $\leq$ which is closed in $X \times X$.

CompOrd: category of compact ordered spaces and order-preserving continuous maps.

Compact ordered spaces

Compact ordered spaces : compact Hausdorff spaces
= Priestley spaces : Boolean spaces.

Compact ordered space $\cong$ Closed subspace of a power of $([0,1], \leq)$.
Compact Hausdorff space $\cong$ Closed subspace of a power of $[0,1]$.
Priestley space $\cong$ Closed subspace of a power of $(\{0,1\}, \leq)$.
Boolean space $\cong$ Closed subspace of a power of $\{0,1\}$.

Dualities with varieties

BooSp	$\xleftrightarrow{\text{op}}$	finitary variety [Stone, 1936]
Priestley	$\xleftrightarrow{\text{op}}$	finitary variety [Priestley, 1970]
CompHaus	$\xleftrightarrow{\text{op}}$	(infinitary) variety [Duskin, 1969]
CompOrd	$\xleftrightarrow{\text{op}}$	???

Open question [Hofmann, Neves and Nora, 2018]

Is the category of compact ordered spaces dually equivalent to a (possibly infinitary) variety?

Characterization of quasivarieties

An object G is said to be

Regular projective if $\hom(G, -)$ preserves regular epim.;

Regular generator if $\hom(G, -)$ reflects regular epim.

$\hom(\mathrm{Free}_1, -) \cong$ forgetful functor

Theorem (Characterization of quasivarieties)

A category is equivalent to a (possibly infinitary) quasivariety iff it is cocomplete and it admits a regular projective regular generator.

Idea: the regular projective regular generator of the statement is the free object Free_1 over a singleton.

Characterization of varieties

Theorem (Characterization of varieties)

A category is equivalent to a (possibly infinitary) variety iff it is equivalent to a quasivariety and internal equivalence relations are effective.

Dualities with varieties

	compl.	reg. inj. reg. cogen.	eff. int. eq. corel.	
BooSp	✓	$\{0,1\}$ (abstr. cofin.)	✓	Dual to fin. variety
Priestley	✓	$\{0,1\}$ (abstr. cofin.)	✓	Dual to fin. variety
CompHaus	✓	$[0,1]$ (not abstr. cofin.)	✓	Dual to inf. variety
CompOrd	✓	$[0,1]$ (not abstr. cofin.)	?	Dual to inf. quasivariety

$\mathsf{CompOrd}^{\mathsf{op}}$ is a variety

Theorem [A. and Reggio, 2020]

Internal equivalence relations in $\mathsf{CompOrd}^{\mathsf{op}}$ are effective.

Theorem

The category $\mathsf{CompOrd}$ of compact ordered spaces is dually equivalent to an (infinitary) variety.

$$\mathsf{CompOrd} \overset{\mathsf{op}}{\cong} \mathbb{SP}([0,1]) = \mathbb{HSP}([0,1])$$

Function symbols of arity a cardinal κ:

order-preserving continuous functions $[0,1]^{\kappa} \to [0,1]$.

$\mathsf{CompOrd}^{\mathsf{op}}$ is Barr-exact. So are the categories of strong proximity lattices and of stably compact frames.

Negative results

Theorem

CompOrd *is not dually equivalent to any finitary variety.*
In fact, CompOrd *is not dually equivalent to*

1. *any finitely accessible category;*
2. *any first-order definable class of structures (no faithful functor* $\mathsf{CompOrd}^{\mathsf{op}} \to \mathsf{Set}$ *preserves directed colimits)* *[Lieberman, Rosický and Vasey, 2019];*
3. *any class of finitary algebras closed under products and subalgebras.*

Finite equational axiomatisation

Does there exist a manageable axiomatization of $\mathsf{CompOrd}^{\mathrm{op}}$?

$\mathsf{CompOrd}^{\mathrm{op}}$ admits a finite equational axiomatisation, i.e. one which uses only finitely many function symbols and finitely many equational axioms.

Which primitive operations?

Idea: take

1. a class $\mathcal{C}$ of order-preserving continuous functions from powers of $[0,1]$ to $[0,1]$ that generate a clone $\mathbf{A} = (A^{[\kappa]})_{\kappa \in \mathrm{Card}}$ on $[0,1]$ such that, for every cardinal κ, every order-preserving continuous function $[0,1]^\kappa \to [0,1]$ is the uniform limit of a sequence in $A^{[\kappa]}$.
2. an order-preserving continuous function $[0,1]^{\mathbb{N}_{>0}} \to [0,1]$ which sends every sequence $(x_1, x_2, \dots)$ satisfying $|x_{n+1} - x_n| \leq \frac{1}{2^n}$ to its limits.

Which classes $\mathcal{C}$ of order-preserving continuous operations on $[0,1]$ satisfy 1?

Closure of clones under uniform limits

Proposition

TFAE for a clone $\mathbf{A} = (A^{[\kappa]})_{\kappa \in \mathrm{Card}}$ of order-preserving continuous functions on $[0,1]$ which contains $\vee$ and $\wedge$.

1. For every cardinal κ, every order-preserving continuous function $[0,1]^\kappa \to [0,1]$ is the uniform limit of a sequence in $A^{[\kappa]}$.
2. 2.1 $A^{[0]}$ is dense in $[0,1]$, and
   2.2 for all $x, y, s, t \in (0,1)$ with $x < y$ there exists $g \in A^{[1]}$ such that $g(x) < s$ and $g(y) > t$.

Primitive operations

Operation	Definition	Why taking them?
$x \vee y$ $x \wedge y$	$:= \max\{x, y\}$ $:= \min\{x, y\}$	To meet the hypothesis of the criterion
$x \oplus y$ $x \odot y$	$:= \min\{x + y, 1\}$ $:= \max\{x + y - 1, 0\}$	To stretch $[0, 1]$ via $x \mapsto x \oplus x, x \mapsto x \odot x$
0 1 $\mathrm{h}(x)$ $\mathrm{j}(x)$	$:= 0$ $:= 1$ $:= \frac{x}{2}$ $:= \frac{1}{2} + \frac{x}{2}$	To obtain a dense subset of $[0, 1]$ (when combined with $\oplus$ and $\odot$)
$\lambda(x_1, x_2, \dots)$	$\approx \lim_{n\to\infty} x_n$ ("$=$" if $\lvert x_{n+1} - x_n\rvert \leq \frac{1}{2^n}$)	To close under unif. lims

Generalisation of Mundici's theorem

In the search of a reasonable set of axioms for $\vee, \wedge, \oplus, \odot, 0, 1$, a generalisation of a theorem by D. Mundici was obtained.

Theorem [Mundici, 1986]

The categories of unital Abelian ℓ-groups and of MV-algebras are equivalent.

Theorem

The categories of *unital commutative distributive ℓ-monoids* and of *MV-monoidal algebras* are equivalent.

Lattice-ordered monoids

Definition

Unital commutative distributive ℓ-monoid: $\langle M; +, \vee, \wedge, 0, 1, -1 \rangle$ s.t.

1. $\langle M; \vee, \wedge \rangle$ is a distributive lattice.
2. $\langle M; +, 0 \rangle$ is a commutative monoid.
3. The operation $+$ distributes over $\vee$ and $\wedge$.
4. $-1 \leq 0 \leq 1$.
5. $-1 + 1 = 0$.
6. $\forall x \in M, \exists n \in \mathbb{N}$ s.t. $n(-1) \leq x \leq n1$.

Example

For X a compact ordered space,

$$\{f \colon X \to \mathbb{R} \mid f \text{ is order-preserving and continuous}\}.$$

Unit interval functor

Given a unital commutative distributive ℓ-monoid $\mathbf{M}$, one equips the set

$$\Gamma(\mathbf{M}) \coloneqq \{x \in M \mid 0 \leq x \leq 1\}$$

with the operations $\vee$, $\wedge$, 0, and 1 by restriction, and

$$x \oplus y \coloneqq (x + y) \wedge 1,$$
$$x \odot y \coloneqq (x + y - 1) \vee 0.$$

MV-monoidal algebras

Definition

MV-monoidal algebra: $\langle A; \oplus, \odot, \vee, \wedge, 0, 1\rangle$ s.t.

1. $\langle A; \vee, \wedge\rangle$ is a distributive lattice.
2. $\langle A; \oplus, 0\rangle$ and $\langle A; \odot, 1\rangle$ are commutative monoids.
3. Both the operations $\oplus$ and $\odot$ distribute over both $\vee$ and $\wedge$.
4. $(x \oplus y) \odot ((x \odot y) \oplus z) = (x \odot (y \oplus z)) \oplus (y \odot z)$.
5. $(x \odot y) \oplus z = ((x \oplus y) \odot ((x \odot y) \oplus z)) \vee z$.
6. $(x \oplus y) \odot z = ((x \odot y) \oplus ((x \oplus y) \odot z)) \wedge z$.

Theorem

The categories of unital commutative distributive ℓ-monoids and of MV-monoidal algebras (with homomorphisms) are equivalent.

Vietoris functor

We have a Vietoris endofunctor

$$\begin{aligned} V\colon \mathsf{CompOrd} &\longrightarrow \mathsf{CompOrd} \\ X &\longmapsto V(X) := \{\text{closed up-sets of } X\}. \end{aligned}$$

Theorem [Hofmann, Neves and Nora, 2018]

The category of coalgebras for $V\colon \mathsf{CompOrd} \to \mathsf{CompOrd}$ is dually equivalent to an (infinitary) quasivariety.

They added to the theory of $\mathbb{SP}([0,1])$ a unary op. $\Diamond$, with

1. $\Diamond 0 = 0$;
2. $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$;
3. for all $t \in [0,1]$, $\Diamond(x \odot t) = \Diamond x \odot t$;
4. $\Diamond(x \odot y) \leq \Diamond x \odot \Diamond y$.

Vietoris functor

Theorem

The category of coalgebras for $V\colon \mathsf{CompOrd} \to \mathsf{CompOrd}$ *is dually equivalent to an (infinitary) variety.*

Future research: obtain a purely categorical proof of the last result.

Recap

- The category of compact ordered spaces is dually equivalent to an (infinitary) variety.
- We have a finite equational axiomatisation (= finitely many operations and equational axioms).
- The employment of operations of infinite arity is necessary.
- En passant, we generalized Mundici's equivalence to unital distributive commutative ℓ-monoids.
- The category of coalgebras for the Vietoris endofunctor on CompOrd is dually equivalent to an (infinitary) variety.

Thank you for your attention.

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