The opposite of the category of compact ordered spaces is monadic over the category of sets

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Outline

Some known duality results

$\mathbf{CompOrd}^{\mathrm{op}}$ is monadic over $\mathbf{Set}$

Vietoris coalgebras

Stone duality for Boolean algebras

Theorem ([Stone, 1936])

$$\textit{Stone spaces} \overset{\mathrm{op}}{\simeq} \textit{Boolean algebras.}$$

Stone space (a.k.a. Boolean space) [Stone, 1936] := compact Hausdorff space with a basis of clopens.

The class of Boolean algebras is a *variety of finitary algebras*, i.e. a class of algebras with operations of finite arity axiomatized by equations (equivalently, closed under $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$). In other words, we have a monadic functor $\mathbf{Boole} \to \mathbf{Set}$ preserving filtered colimits.

Duality for compact Hausdorff spaces

$$\mathbf{Stone} \hookrightarrow \mathbf{CompHaus}.$$

Theorem ([Duskin, 1969])

The opposite of the category of compact Hausdorff spaces is monadic over $\mathbf{Set}$.

$\mathbf{C}$ is monadic over $\mathbf{Set}$ $\iff$ $\mathbf{C}$ is equivalent to a variety of (possibly infinitary) algebras.

$\mathbf{CompHaus}^{\mathrm{op}}$ is equivalent to a variety of possibly infinitary algebras.

(A *variety of (possibly infinitary) algebras* is a class of algebras in a (possibly large, possibly infinitary) signature with free algebras axiomatized by equations (equivalently, closed under $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$).)

Negative finitary result

Theorem

$\mathbf{CompHaus}$ *is* *<u>not</u>* *dually equivalent to any variety of* *<u>finitary</u>* *algebras.*

Deducible e.g. from [Gabriel and Ulmer, 1971], strenghtened by [Bankston, 1982], [Banaschewski, 1983], [Banaschewski, 1984], [Rosický, 1989], [Marra and Reggio, 2017], [Lieberman, Rosický and Vasey, 2019].

Adding the order

Theorem ([Priestley, 1970])

Priestley spaces $\overset{\mathrm{op}}{\simeq}$ *bounded distributive lattices.*

Priestley space [Priestley, 1970] $:=$ Stone space equipped with a partial order $\leq$, satisfying a condition called "total order-disconnectedness": if $x \not\leq y$, then there exists a clopen up-set U of X such that $x \in U$ and $y \notin U$.

Bounded distributive lattices form a variety of finitary algebras.

The picture

$$\begin{tikzcd}
\text{Stone spaces} \arrow[r, hook] \arrow[d, hook] & \text{Priestley spaces} \arrow[d, hook] \\
\text{Compact Hausdorff spaces} \arrow[r, hook] & \text{Compact ordered spaces}
\end{tikzcd}$$

Compact ordered spaces

Definition ([Nachbin, 1948])

A *compact ordered space* is a compact Hausdorff space X with a partial order $\leq$ which is closed in $X \times X$; in other words, if $(x_i)_{i\in I} \to x$, $(y_i)_{i\in I} \to y$, and $x_i \leq y_i$, then $x \leq y$.

$\mathbf{CompOrd}$: category of compact ordered spaces and order-preserving continuous maps.

Examples

- $[0, 1]$ with Euclidean topology and $\leq$.
- Compact Hausdorff space with identity relation. $\mathbf{CompHaus} \hookrightarrow \mathbf{CompOrd}$.
- Priestley space. $\mathbf{Priestley} \hookrightarrow \mathbf{CompOrd}$.

Dualities with varieties

Open question in [Hofmann, Neves and Nora, 2018]

Is the opposite of the category of compact ordered spaces monadic over $\mathbf{Set}$?

Partial result in [Hofmann, Neves and Nora, 2018]: $\mathbf{CompOrd}$ is dually equivalent to an $\aleph_1$-ary quasi-variety.

($\aleph_1$-ary *quasi*-variety: function symbols of at most countable arity, quasi-equations with at most countably many premises.)

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Effectiveness

The missing piece to prove that $\mathbf{CompOrd}^{\mathrm{op}}$ is monadic over $\mathbf{Set}$:

Theorem ([A. and Reggio, 2020])

Every equivalence relation in $\mathbf{CompOrd}^{\mathrm{op}}$ is effective.

The proof is more complicated than that for $\mathbf{CompHaus}$.

Corollary ([A., 2019], [A. and Reggio, 2020])

The opposite of the category of compact ordered spaces is monadic over $\mathbf{Set}$.

Purely categorical proof (no equations around).

Monadic functor:

$$\hom_{\mathbf{CompOrd}}(-,[0,1])\colon \mathbf{CompOrd}^{\mathrm{op}} \to \mathbf{Set}.$$

Negative finitary result

Theorem

$\mathbf{CompOrd}$ *is not dually equivalent to any variety of finitary algebras.*

More is true: no faithful functor $\mathbf{CompOrd}^{\mathrm{op}} \to \mathbf{Set}$ preserves codirected limits [Lieberman, Rosický and Vasey, 2019].

Dualities

$$
\begin{aligned}
\mathbf{Stone} &\overset{\mathrm{op}}{\simeq} \text{variety of finitary algebras [Stone, 1936]},\\
\mathbf{Priestley} &\overset{\mathrm{op}}{\simeq} \text{variety of finitary algebras [Priestley, 1970]},\\
\mathbf{CompHaus} &\overset{\mathrm{op}}{\simeq} \text{variety of infinitary algebras [Duskin, 1969]},\\
\mathbf{CompOrd} &\overset{\mathrm{op}}{\simeq} \text{variety of infinitary algebras [A., 2019], [A. and Reggio, 2020]}.
\end{aligned}
$$

Outline

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Vietoris coalgebras

Purposes

Vietoris V for Stone, Priestley, compact Hausdorff and compact ordered spaces.

Is $\mathbf{CoAlg}(V)^{\mathrm{op}}$ monadic over $\mathbf{Set}$?

Vietoris functor

The Vietoris construction [Vietoris, 1922] associates to a compact Hausdorff space X the set $V(X)$ of closed subspaces of X, appropriately topologized; $V(X)$ is compact Hausdorff, as well.

The classical *Vietoris* endofunctor $V_{\mathbf{CompHaus}} \colon \mathbf{CompHaus} \to \mathbf{CompHaus}$ maps a compact Hausdorff space X to the space of closed subsets of X, with an appropriate topology.

$V_{\mathbf{CompHaus}} \colon \mathbf{CompHaus} \to \mathbf{CompHaus}$ restricts to $V_{\mathbf{Stone}} \colon \mathbf{Stone} \to \mathbf{Stone}$.

Vietoris for Priestley spaces

- The (convex-)*Vietoris* functor $V_{\mathbf{Priestley}} : \mathbf{Priestley} \to \mathbf{Priestley}$ on Priestley spaces associates to a Priestley space X the set of order-convex closed subsets of X, with the Egli-Milner order, and an appropriate topology.

  $V_{\mathbf{Priestley}}$ restricts to $\mathbf{Stone}$.

- The *lower Vietoris* functor $L_{\mathbf{Priestley}} : \mathbf{Priestley} \to \mathbf{Priestley}$ on Priestley spaces associates to a Priestley space X the set of closed up-sets of X equipped with the reverse inclusion order, and an appropriate topology.

  $L_{\mathbf{Priestley}}$ does <u>not</u> restrict to $\mathbf{Stone}$.

Vietoris for compact ordered spaces

- The (convex-)*Vietoris* functor $V_{\mathbf{CompOrd}} \colon \mathbf{CompOrd} \to \mathbf{CompOrd}$ for compact ordered spaces associates to a compact ordered space X the compact ordered space of order-convex closed subsets of X, with the Egli-Milner order, and an appropriate topology.

  $V_{\mathbf{CompOrd}}$ restricts to $\mathbf{CompHaus}$.

- The *lower Vietoris* functor $L_{\mathbf{CompOrd}} \colon \mathbf{CompOrd} \to \mathbf{CompOrd}$ for compact ordered spaces associates to a compact ordered space X the compact ordered space of closed up-sets of X, with the reverse inclusion order order, and an appropriate topology.

  $L_{\mathbf{CompOrd}}$ does <u>not</u> restrict to $\mathbf{CompHaus}$.

Coalgebras

A *coalgebra* for an endofunctor $F\colon \mathbf{C} \to \mathbf{C}$ is given by (an object X of $\mathbf{C}$ and) a morphism $X \to F(X)$.

The picture

$$
\begin{tikzcd}
\mathbf{CoAlg}(V_{\mathbf{Stone}}) \arrow[r, hook] \arrow[d, hook] & \mathbf{CoAlg}(V_{\mathbf{Priestley}}) \arrow[d, hook] & \mathbf{CoAlg}(L_{\mathbf{Priestley}}) \arrow[d, hook] \\
\mathbf{CoAlg}(V_{\mathbf{CompHaus}}) \arrow[r, hook] & \mathbf{CoAlg}(V_{\mathbf{CompOrd}}) & \mathbf{CoAlg}(L_{\mathbf{CompOrd}})
\end{tikzcd}
$$

Top row: known to be dually equivalent to some varieties of finitary algebras.
Our main result + [Hofmann, Neves and Nora, 2018] $\Rightarrow$

Theorem

The opposite of the category of coalgebras for the lower Vietoris functor on compact ordered spaces is monadic over $\mathbf{Set}$.

Some questions

$$
\begin{tikzcd}
\mathbf{CoAlg}(V_{\mathbf{Stone}}) \arrow[r, hook] \arrow[d, hook] & \mathbf{CoAlg}(V_{\mathbf{Priestley}}) \arrow[d, hook] & \mathbf{CoAlg}(L_{\mathbf{Priestley}}) \arrow[d, hook] \\
\mathbf{CoAlg}(V_{\mathbf{CompHaus}}) \arrow[r, hook] & \mathbf{CoAlg}(V_{\mathbf{CompOrd}}) & \mathbf{CoAlg}(L_{\mathbf{CompOrd}})
\end{tikzcd}
$$

1. Prove purely categorically that $\mathbf{CoAlg}(L_{\mathbf{CompOrd}})^{\mathrm{op}}$ is monadic over $\mathbf{Set}$.
2. Is the opposite of $\mathbf{CoAlg}(V_{\mathbf{CompOrd}})$ monadic over $\mathbf{Set}$?
3. Is the opposite of $\mathbf{CoAlg}(V_{\mathbf{CompHaus}})$ monadic over $\mathbf{Set}$? (Already known?)

Recap

Main results:

1. $\mathbf{CompOrd}^{\mathrm{op}}$ is monadic over $\mathbf{Set}$.
2. $\mathbf{CoAlg}(L_{\mathbf{CompOrd}})^{\mathrm{op}}$ is monadic over $\mathbf{Set}$.

Questions:

1. Prove purely categorically that $\mathbf{CoAlg}(L_{\mathbf{CompOrd}})^{\mathrm{op}}$ is monadic over $\mathbf{Set}$.
2. Is the opposite of $\mathbf{CoAlg}(V_{\mathbf{CompOrd}})$ monadic over $\mathbf{Set}$?
3. Is the opposite of $\mathbf{CoAlg}(V_{\mathbf{CompHaus}})$ monadic over $\mathbf{Set}$? (Already known?)

Thank you for your attention.

References I

Abbadini, M. (2019).
The dual of compact ordered spaces is a variety.
Theory Appl. Categ. 34(44):1401–1439.

Abbadini, M., and Reggio, L. (2020).
On the axiomatisability of the dual of compact ordered spaces.
Appl. Categ. Struct. 28(6):921–934.

Banaschewski, B. (1983).
On categories of algebras equivalent to a variety.
Algebra Universalis, 16(2):264–267.

Banaschewski, B. (1984)
More on compact Hausdorff spaces and finitary duality.
Canad. J. Math. 36(6):1113-–1118.

References II

Bankston, P. (1982).
Some obstacles to duality in topological algebra.
Canadian J. Math., 34(1):80–90.

Duskin, J. (1969).
Variations on Beck's tripleability criterion.
In Mac Lane, S., editor, *Reports of the Midwest Category Seminar, III*, pages 74–129. Springer, Berlin.

Gabriel, P. and Ulmer, F. (1971).
Lokal präsentierbare Kategorien, volume 221 of *Lecture Notes in Mathematics*.
Springer-Verlag, Berlin-New York.

Hofmann, D., Neves, R., and Nora, P. (2018).
Generating the algebraic theory of $C(X)$: the case of partially ordered compact spaces.
Theory Appl. Categ., 33:276–295.

References III

Lieberman, M., Rosický, J., and Vasey, S. (2019).
Hilbert spaces and C^*-algebras are not finitely concrete.
Preprint available at `arXiv:1908.10200`.

Marra, V. and Reggio, L. (2017).
Stone duality above dimension zero: axiomatising the algebraic theory of $\mathrm{C}(X)$.
Adv. Math., 307:253–287.

Nachbin, L. (1948).
Sur les espaces topologiques ordonnés.
C. R. Acad. Sci. Paris, 226:381–382.

Priestley, H. A. (1970).
Representation of distributive lattices by means of ordered Stone spaces.
Bull. London Math. Soc., 2:186–190.

References IV

Rosický, J. (1989).
Elementary categories.
Arch. Math. (Basel), 3:284–288.

Stone, M. H. (1936).
The theory of representations for Boolean algebras.
Trans. Amer. Math. Soc., 40(1):37–111.

Vietoris, L. (1922)
Bereiche zweiter Ordnung.
Monatsh. Math. Phys., 32(1):258–280.