Priestley duality above dimension zero

Algebraic axiomatisability of the dual of compact ordered spaces

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Context: dualities

Dualities

Stone duality for Boolean Algebras (Stone, 1936)

Boolean algebras	*Boolean spaces* (Comp. Hausd. 0-dimensional)

Priestley duality (Priestley, 1970)

(Bounded) distributive lattices	*Priestley spaces* (Bool. space with a partial order + total order-disconnectedness)

Dualities: above 0-dimensionality

Duality for compact Hausdorff spaces (Duskin, 1969)

A variety of (infinitary) algebras — *Compact Hausdorff spaces*

Duality for compact ordered spaces

A variety of (possibly infinitary) algebras?
(Yes)

Compact ordered spaces

Compact ordered spaces

Compact ordered space

Definition (Nachbin, 1965)
A *compact ordered space* is a compact Hausdorff space X, equipped with a partial order $\leq$, which is closed in $X \times X$ with respect to the product topology.

A *morphism of compact ordered spaces* is a continuous monotone map.

Example
$[0, 1]$.

Example
$[0, 1]^\kappa$.

Example
$Y \subseteq [0, 1]^\kappa$, Y closed.

Every compact ordered space is isomorphic to a closed subspace of a power of $[0, 1]$.

Comparison

Compact ordered space = closed subspace of a power of $[0, 1]$.

Compact Hausdorff space = closed subspace of a power of $[0, 1]$.

Priestley space = closed subspace of a power of $\{0, 1\}$.

Boolean space = closed subspace of a power of $\{0, 1\}$.

The dual of compact ordered spaces

The dual of compact ordered spaces

$\mathsf{KH}_{\leq}$:= category of compact ordered spaces.

$\mathsf{KH}_{\leq}$ is dually equivalent to a quasi-variety of (infinitary) algebras (Hofmann, Neves and Nora, 2018).

Characterisation of quasi-varieties
A category C *is equivalent to a quasi-variety of (possibly infinitary) algebras if, and only if,* C *is locally small and*

1. C *is cocomplete;*
2. C *has a regular projective regular generator.*

$\mathsf{KH}_{\leq}$ is dually equivalent to a quasi-variety because it is locally small, complete and it has a regular injective regular cogenerator: $[0, 1]$.

The dual of compact ordered spaces

Question (Hofmann, Neves, Nora, 2018)
Is the category of compact ordered spaces also dually equivalent to a variety of (possibly infinitary) algebras?

Main result
The category of compact ordered spaces is dually equivalent to a variety of infinitary algebras.

(It is necessary to resort to infinitary operations.)

Recap

	Comp. Hausd. 0-dim.	Comp. Hausd.
Without order	Boolean spaces Dual: finitary variety	Comp. Hausd. spaces Dual: infinitary variety
With order	Priestley spaces Dual: finitary variety	Compact ordered spaces Dual: infinitary variety

Dualities

	Bool. sp.	*Priest. sp.*	*Comp. Hausd.*	*Comp. ord.*
Dual	Finit. var. (Bool. alg.)	Finit. var. (Distr. latt.)	Infin. var.	Infin. var.
κ-ary terms	$\{0,1\}^\kappa \to \{0,1\}$ cont.	$\{0,1\}^\kappa \to \{0,1\}$ cont. monot.	$[0,1]^\kappa \to [0,1]$ cont.	$[0,1]^\kappa \to [0,1]$ cont. monot.
Gener. by	$\{0,1\}$	$\{0,1\}$	$[0,1]$	$[0,1]$

In the variety which is dual to compact ordered spaces:

Terms of arity κ: continuous monotone functions $[0,1]^\kappa \to [0,1]$.

Axioms: the equations that hold in $[0,1]$.

Sketch of proof of main result

Quasi-varieties and varieties

To be proven
The category of compact ordered spaces is dually equivalent to a variety of (possibly infinitary) algebras.

It was already observed that $\mathsf{KH}_{\leq}^{\mathrm{op}}$ is equivalent to a quasi-variety.

What is missing to prove that $\mathsf{KH}_{\leq}^{\mathrm{op}}$ is a variety?

Theorem
A quasi-variety C *is a variety if, and only if, every (internal) equivalence relation of* C *is effective.*

Corelations

A relation on an object X in a given category is an (equivalence class of) monomorphisms $r\colon R \rightarrowtail X \times X$.

A corelation on an object X is an epimorphism $r\colon X + X \twoheadrightarrow R$.

We encode an epimorphism $f\colon X \twoheadrightarrow Y$ of compact ordered spaces (=surjective morphism) internally of X, via

$$\preccurlyeq_f := \{(x, y) \in X \times X \mid f(x) \leq f(y)\},$$

which is a preorder on X which extends the partial order $\leq$ on X, and which is closed in $X \times X$.

Corelations

A corelation $r\colon X+X \twoheadrightarrow R$ is encoded by the preorder $\preccurlyeq_r$ on $X+X$.

We rephrase the conditions of coreflexivity, cosymmetry, cotransitivity and coeffectiveness as order-topological properties on $\preccurlyeq_r$.

Proposition
If $\preccurlyeq_r$ encodes a coreflexive cosymmetric cotransitive corelation, $\preccurlyeq_r$ encodes a coeffective corelation.

This proves that equivalence relations in $\mathsf{KH}_{\leq}^{\mathrm{op}}$ are effective, and thus that the quasi-variety $\mathsf{KH}_{\leq}^{\mathrm{op}}$ is actually a variety.

Negative results

Negative results

Theorem
The category of compact ordered spaces is not dually equivalent to any variety of finitary algebras.

More generally:

Theorem
Let C *be a full subcategory of* $\mathsf{KH}_{\leq}$, *with* $\mathsf{C} \supseteq \mathsf{Priestley}$. *If the dual of* C *is equivalent to a finitely accessible category, then* $\mathsf{C} = \mathsf{Priestley}$.

Theorem (Suggested by Lieberman, Rosický and Vasey, 2019)
The category of compact ordered spaces is not dually equivalent to any elementary class of structures.

Conclusions

Conclusions

Main result
The opposite of the category of compact ordered spaces is equivalent to a variety of infinitary algebras.

Sketch of the proof.
The category of compact ordered spaces satisfies:

1. Completeness.
2. Existence of a regular injecive regular cogenerator object ($[0,1]$).
3. Coequivalence corelations are coeffective.

□

κ-ary terms: $[0,1]^\kappa \to [0,1]$ continuous and monotone.

Axioms: the equations that hold in $[0,1]$.

It is necessary to resort to infinitary operations.

Thank you for your attention!