Archimedean Cauchy-complete MV-algebras form a variety

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Archimedean MV-algebras are MV-algebras which are isomorphic to a subalgebra of $[0,1]^X$, for some set X.

Archimedean MV-algebras may be endowed with a canonical metric d, which, in any representation $\iota \colon A \hookrightarrow [0,1]^X$, coincides with the supremum distance.

We denote by $\mathrm{MV}_{\mathrm{AC}}$ the category with

1. objects: archimedean Cauchy-complete MV-algebras;
2. morphisms: MV-homomorphisms.

Theorem (Main result)

$\mathrm{MV}_{\mathrm{AC}}$ *is a variety of algebras.*

(We admit operations of infinite arity.)

Definition

Let (X, d) be a metric space. A sequence $(x_n)_{n \in \mathbb{N} = \{1,2,3\ldots,\}}$ is said to be *super-Cauchy* if, for every $n \in \mathbb{N}$, we have $d(x_n, x_{n+1}) \leqslant \frac{1}{2^{n+1}}$.

Lemma

Let (X, d) be a metric space. Then (X, d) is complete if, and only if, every super-Cauchy sequence converges.

Idea

Add an operation γ to the set of MV-operations of countably infinite arity, together with some new axioms, so that

1. any model is an archimedean MV-algebra,
2. γ maps super-Cauchy sequences to their limit.

The intended interpretation of γ is of the following form.

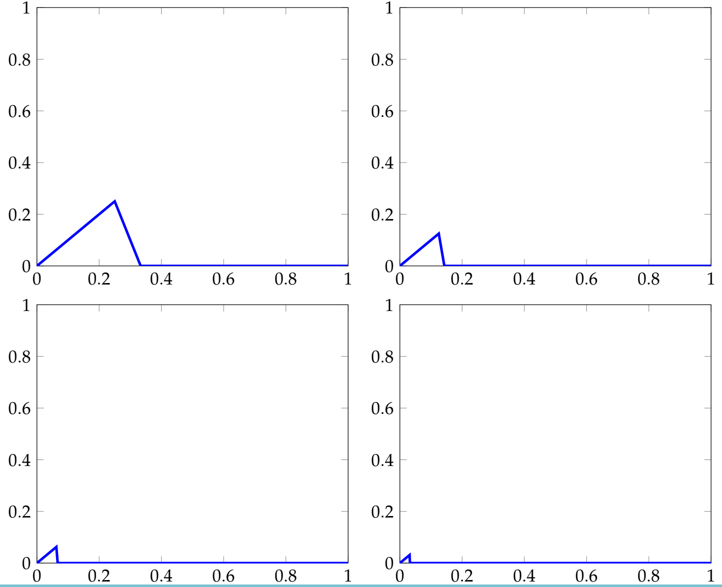
$$\gamma(a_1, a_2, a_3, \dots) = \lim_{n\to\infty} \rho_n(a_1, \dots, a_n)$$

where

1. for every $n \in \mathbb{N}$, ρ_n is an MV-algebraic term,
2. $(\rho_n(a_1, \dots, a_n))_{n\in\mathbb{N}}$ is a super-Cauchy sequence,
3. If $(a_n)_{n\in\mathbb{N}}$ is a super-Cauchy sequence, then, for all $n \in \mathbb{N}$, $\rho_n(a_1, \dots, a_n) = a_n$.

If we find a sequence of MV-algebraic terms $(\rho_n)_{n\in\mathbb{N}}$ that satisfies (1), (2), (3), then γ is well-defined on any archimedean Cauchy-complete MV-algebra, and maps super-Cauchy sequences to their limit.

For $n \in \mathbb{N}$, define the MV-algebraic term
$\theta_n(x) := x \wedge (1 - (2^{n+1} - 1)x)$.



Important properties of θ_n.
If A is an archimedean MV-algebra, we have

1. for every $x \in A$,
$$\|\theta_n(x)\| \leqslant \frac{1}{2^{n+1}};$$
2. for every $x \in A$ such that $\|x\| \leqslant \frac{1}{2^{n+1}}$,
$$\theta_n(x) = x.$$

Let A be an archimedean MV-algebra. For any $a_1, a_2 \in A$, we have

$$a_2 = (a_1 \oplus (a_2 \ominus a_1)) \ominus (a_1 \ominus a_2).$$

What happens if we consider a slight modification of the right-hand side?

$$\rho_2(a_1, a_2) \coloneqq (a_1 \oplus \theta_1(a_2 \ominus a_1)) \ominus \theta_1(a_1 \ominus a_2).$$

$$\rho_2(a_1, a_2) := (a_1 \oplus \theta_1(a_2 \ominus a_1)) \ominus \theta_1(a_1 \ominus a_2).$$

1. If $\mathrm{d}(a_1, a_2) \leqslant \frac{1}{4}$, then $\rho_2(a_1, a_2) = a_2$.
2. In general, $\mathrm{d}(\rho_2(a_1, a_2), a_1) \leqslant \frac{1}{4}$.

$$\rho_1(a_1) \coloneqq a_1,$$

$$\rho_{n+1}(a_1, \ldots, a_{n+1}) \coloneqq (\rho_n(a_1, \ldots, a_n) \oplus \theta_n(a_{n+1} \ominus a_n)) \ominus \theta_n(a_n \ominus a_{n+1}).$$

1. For every $n \in \mathbb{N}$, ρ_n is an MV-algebraic term.
2. $(\rho_n(a_1, \ldots, a_n))_{n \in \mathbb{N}}$ is a super-Cauchy sequence.
3. If $(a_n)_{n \in \mathbb{N}}$ is a super-Cauchy sequence, then, for all $n \in \mathbb{N}$, $\rho_n(a_1, \ldots, a_n) = a_n$.

Then, in any archimedean Cauchy-complete MV-algebra,

$$\gamma(a_1, a_2, a_3, \ldots) = \lim_{n \to \infty} \rho_n(a_1, \ldots, a_n)$$

is well-defined, and maps super-Cauchy sequences to their limit.

The variety $\widetilde{\mathrm{MV}}_{\mathrm{AC}}$

Operations

Operations of MV-algebras, together with an operation γ of countably infinite arity.

Axioms

0. Axioms of MV-algebra.
1. $\gamma(a, a, a, \dots) = a$.
2. $\gamma(\theta_1(a), \theta_2(a), \theta_3(a), \dots) = 0$.
3. (For each $n \in \mathbb{N}$) $\mathrm{d}(\gamma(a_1, a_2, a_3, \dots), \rho_n(a_1, \dots, a_n)) \leqslant \frac{1}{2^n}$.

Every archimedean Cauchy-complete MV-algebra satisfies the axioms.

Lemma

The axioms

1. $\gamma(a, a, a, \dots) = a$;
2. $\gamma(\theta_1(a), \theta_2(a), \theta_3(a), \dots) = 0$.

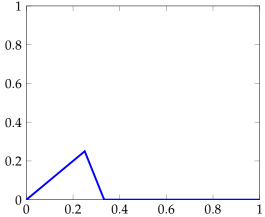
imply the archimedean property.

Proof.

Let a be infinitesimal. Then

$$0 = \gamma(\theta_1(a), \theta_2(a), \theta_3(a), \dots) = \gamma(a, a, a, \dots) = a.$$

□



The axiom

3. (For each $n \in \mathbb{N}$) $\mathrm{d}(\gamma(a_1, a_2, a_3, \dots), \rho_n(a_1, \dots, a_n)) \leqslant \frac{1}{2^n}$.

implies Cauchy-completeness, and 'defines' γ as the limit of $(\rho_n)_{n \in \mathbb{N}}$.

Let $U\colon \widetilde{\mathrm{MV}}_{\mathrm{AC}} \to \mathbb{MV}$ be the forgetful functor from the variety $\widetilde{\mathrm{MV}}_{\mathrm{AC}}$ to the variety $\mathbb{MV}$ of MV-algebras.

Theorem

1. *U is full and faithful.*
2. *U is injective on objects.*
3. *The image of U on objects coincides with the class of archimedean Cauchy-complete MV-algebras.*

Conclusion

Recall: $\mathrm{MV}_{\mathrm{AC}}$ is the category with

1. objects: archimedean Cauchy-complete MV-algebras;
2. morphisms: MV-homomorphisms.

Theorem (Main result)

$\mathrm{MV}_{\mathrm{AC}}$ *is a variety of algebras.*

Thank you for your attention.