Operations that preserve integrability, and truncated Riesz spaces

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Talk based on
M. Abbadini, *Operations that preserve integrability, and truncated Riesz spaces*, arXiv:1807.05533

BLAST 2018

University of Denver, Colorado, USA

Overview

Part I: Operations that preserve integrability
We characterize the operations under which the $\mathcal{L}^1$ spaces are closed. We exhibit a simple set of generating operations.

Part II: Truncated Riesz spaces
We investigate the equational laws satisfied by the operations of Part I. We obtain an explicit axiomatization of the infinitary variety generated by $\mathcal{L}^1$ spaces with these operations. We obtain a representation theorem for free objects in the variety.

Overview

Part I: Operations that preserve integrability

Part II: Truncated Riesz spaces

For $(\Omega, \mathcal{F}, \mu)$ a measure space, where we allow $\mu(\Omega) = \infty$, we say that a function $f : \Omega \to \mathbb{R}$ is *integrable* if it is $\mathcal{F}$-measurable and such that $\int_\Omega |f| \, d\mu < \infty$. Let us set

$$\mathcal{L}^1(\mu) = \{f : \Omega \to \mathbb{R} \mid f \text{ is integrable}\}.$$

If $f, g \in \mathcal{L}^1(\mu)$, then

- $f + g \in \mathcal{L}^1(\mu)$;
- $f \cdot g$ may fail to belong to $\mathcal{L}^1(\mu)$.

We say that $\mathcal{L}^1(\mu)$ is *closed under* the operation $+ : \mathbb{R}^2 \to \mathbb{R}$, but may fail to be closed under the operation $\cdot : \mathbb{R}^2 \to \mathbb{R}$.

The notion for a general operation $\tau : \mathbb{R}^I \to \mathbb{R}$ is as follows.

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The notion for a general operation $\tau : \mathbb{R}^I \to \mathbb{R}$ is as follows.

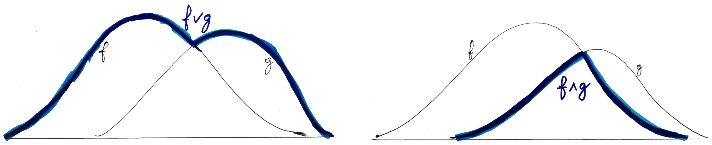
For I a set and $\tau\colon \mathbb{R}^I \to \mathbb{R}$, we say $\mathcal{L}^1(\mu)$ *is closed under* τ if, for all $(f_i)_{i\in I} \subseteq \mathcal{L}^1(\mu)$, the function

$$\begin{aligned}\tau\big((f_i)_{i\in I}\big)\colon \Omega &\longrightarrow \mathbb{R}\\ \omega \in \Omega &\longmapsto \tau\big((f_i(\omega))_{i\in I}\big)\end{aligned}$$

belongs to $\mathcal{L}^1(\mu)$. In such case, we also say τ *preserves integrability over* μ.

Examples of operations that preserve integrability over every measure

1. The binary addition $+ : \mathbb{R}^2 \to \mathbb{R}$.
2. For $\lambda \in \mathbb{R}$, the multiplication $\lambda(\,\cdot\,) : \mathbb{R} \to \mathbb{R}$ by λ.
3. The element $0 \in \mathbb{R}$.
4. The binary sup $\vee : \mathbb{R}^2 \to \mathbb{R}$ and inf $\wedge : \mathbb{R}^2 \to \mathbb{R}$.

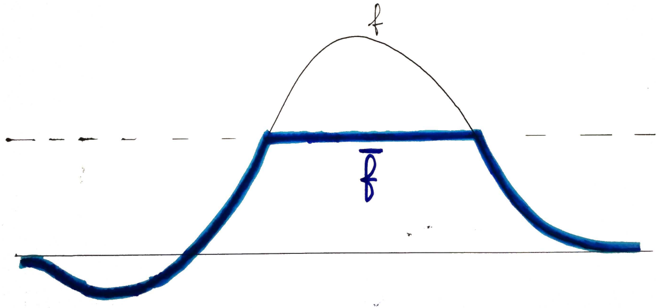


Examples of operations that preserve integrability over every measure

5. The unary operation

$$\begin{aligned} \overline{\cdot} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \overline{x} := x \wedge 1, \end{aligned}$$

called *truncation*.

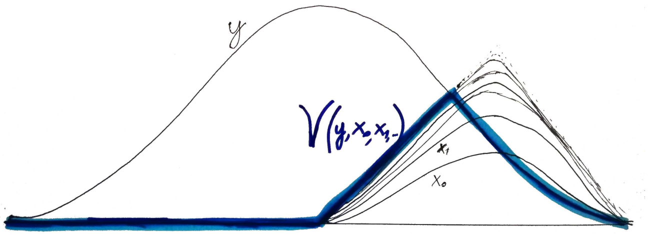


Examples of operations that preserve integrability over every measure

6. The operation of countably infinite arity $\curlyvee : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$:

$$\curlyvee(y, x_0, x_1, x_2, \dots) := \sup_{n \in \mathbb{N}} \{x_n \wedge y\},$$

called *truncated supremum*.



Question

Under which operations $\mathbb{R}^I \to \mathbb{R}$ are all $\mathcal{L}^1$ spaces closed? Equivalently, which operations preserve integrability over every measure?

Theorem

The operations that preserve integrability over every measure are exactly those obtained by composition from

$$+,\ \lambda(\,\cdot\,)\ \textit{(for each } \lambda \in \mathbb{R}\textit{)},\ 0,\ \vee,\ \wedge,\ \overline{\,\cdot\,}\ \textit{and}\ \curlyvee.$$

There is an explicit characterization of the operations $\mathbb{R}^I \to \mathbb{R}$ that preserve integrability over every measure.

Finite Arity

$\tau \colon \mathbb{R}^n \to \mathbb{R}$ preserves integrability over every measure if, and only if,

1. τ is Borel measurable, and
2. $\exists \lambda_0, \dots, \lambda_{n-1} \in \mathbb{R}$ such that, for every $x_0, \dots, x_{n-1} \in \mathbb{R}$, we have

$$|\tau(x_0, \dots, x_{n-1})| \leqslant \lambda_0 |x_0| + \cdots + \lambda_{n-1} |x_{n-1}|.$$

Overview

Part I: Operations that preserve integrability

Part II: Truncated Riesz spaces

Idea (R.N. Ball)

For $f \in \mathcal{L}^1(\mu)$,

$$\bar{f} := f \wedge 1 \in \mathcal{L}^1(\mu),$$

even if $1 \notin \mathcal{L}^1(\mu)$.

Therefore a "truncation" operation is defined even in the absence of a weak unit.

Definition

A *truncated Riesz space* is a Riesz space E that is endowed with a unary operation $\overline{\cdot}\colon E \to E$, called *truncation*, which has the following properties.

(T1) For all $f \in E$, $\left(\overline{f}\right)^{-} = f^{-}$, and $\left(\overline{f}\right)^{+} = \overline{f^{+}}$.

(T2) For all $f, g \in E^{+}$, we have $f \wedge \overline{g} \leqslant \overline{f} \leqslant f$.

(T3) For all $f \in E^{+}$, if $\overline{f} = 0$, then $f = 0$.

(T4) For all $f \in E^{+}$, if $nf = \overline{nf}$ for every $n \in \mathbb{N}$, then $f = 0$. □

Based on R.N. Ball, *Truncated abelian lattice-ordered groups I: The pointed (Yosida) representation*, Topology Appl., 162, 2014, pp. 43–65.

We will see that the operations that preserve integrability are related to the *category of Dedekind σ-complete truncated Riesz spaces* (whose morphisms are the Riesz morphisms which preserve the existing countable suprema and the truncation).

Theorem

The category of Dedekind σ-complete truncated Riesz spaces is an infinitary variety of algebras.

Primitive operations:

1. Primitive operations of Riesz spaces: $+, \lambda(\,\cdot\,)$(for each $\lambda \in \mathbb{R}$), $0, \vee, \wedge$.
2. Truncation $\overline{\cdot}$.
3. Operation of countably infinite arity $\curlyvee$:

$$\curlyvee(y, x_0, x_1, x_2, \dots) := \sup_{n\in\mathbb{N}}\{x_n \wedge y\}.$$

Axioms:

Axioms of Riesz spaces + finitely many additional ones.

$$\left(\mathbb{R}, \left\{+, \lambda(\,\cdot\,)(\text{for each } \lambda \in \mathbb{R}), 0, \vee, \wedge, \bar{\cdot}, \bigvee\right\}\right)$$

is a Dedekind σ-complete truncated Riesz space.

Theorem

The variety of Dedekind σ-complete truncated Riesz spaces is

$$\mathsf{HSP}\left(\mathbb{R}, \left\{+, \lambda(\,\cdot\,)(\textit{for each } \lambda \in \mathbb{R}), 0, \vee, \wedge, \overline{\cdot}, \curlyvee\right\}\right).$$

Sketch of proof.

Starting point: Loomis-Sikorski Theorem for Riesz spaces, i.e. embedding of an archimedean Riesz space into $\frac{\mathbb{R}^X}{\mathcal{I}}$, with all existing countable suprema preserved (e.g. G. Buskes, A. Van Rooij, *Representation of Riesz spaces without the Axiom of Choice*, Nepali Math. Sci. Rep., 16(1-2):19-22, 1997.).

We make an adaptation for truncated Riesz spaces. □

Stronger result

Every quasi-equation with countably many premises that holds in $\mathbb{R}$ holds in every Dedekind σ-complete truncated Riesz space.

Theorem

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Corollary

The free Dedekind σ-complete truncated Riesz space is given by

$$\mathsf{Free}_I := \{\tau\colon \mathbb{R}^I \to \mathbb{R} \mid \tau \textit{ preserves integrability over every measure}\}.$$

Finite Arity

$$\mathsf{Free}_n = \{\tau\colon \mathbb{R}^n \to \mathbb{R} \mid \tau \text{ is Borel measurable, } \exists \lambda_0, \ldots, \lambda_{n-1} \in \mathbb{R} : \forall x_0, \ldots, x_{n-1} \in \mathbb{R} \ \ |\tau(x_0, \ldots, x_{n-1})| \leqslant \lambda_0 |x_0| + \cdots + \lambda_{n-1} |x_{n-1}|\}.$$

Finite measures and weak units

We have obtained analogous results in the case that μ is a finite measure (i.e. $\mu(\Omega) < \infty$).

If μ is finite, the constant function 1 belongs to, and is a weak unit of, $\mathcal{L}^1(\mu)$.

Theorem

The operations that preserve integrability over every finite measure are exactly those obtained by composition from

$$+,\ \lambda(\,\cdot\,)\ (\text{for each } \lambda \in \mathbb{R}),\ 0,\ \vee,\ \wedge,\ \curlyvee \text{ and } 1.$$

Corresponding (infinitary) variety: Dedekind σ-complete Riesz spaces with weak unit. Representation of free objects.

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Further research: From measurability to integration (with V. Marra)

We consider the operator

$$\begin{aligned} \int \colon \mathcal{L}^1(\mu) &\longrightarrow \mathbb{R} \\ f &\longmapsto \int f \,\mathrm{d}\mu \in \mathbb{R} \end{aligned}$$

as an operation of a 2-sorted variety.

(Inspired by the work of T. Kroupa and V. Marra, who studied this idea in the case of finite additivity, instead of σ-additivity.)

The first sort has the axioms of Dedekind σ-complete truncated Riesz spaces. Second sort? Axiomatization? Generating objects? Free objects?

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Thank you for your attention.