Equivalence à la Mundici for lattice-ordered monoids

Marco Abbadini

Nonclassical Logic Seminar, 8 May 2020

Università degli Studi di Milano

Mundici's equivalence

Mundici: the category of unital Abelian lattice-ordered groups is equivalent to the category of MV-algebras.

This establishes a bridge between

$$C(X, \mathbb{R}) := \{f : X \to \mathbb{R} \text{ continuous}\}$$

and

$$C_{\leq}(X, [0,1]) := \{f : X \to [0,1] \text{ continuous}\}$$

where X is a compact space.

Algebras of continuous monotone functions

We want to establish a bridge between

$$C_{\leq}(X, \mathbb{R}) := \{f \colon X \to \mathbb{R} \text{ continuous and monotone}\}$$

and

$$C_{\leq}(X, [0,1]) := \{f \colon X \to [0,1] \text{ continuous and monotone}\}$$

where X is a compact space, endowed with a partial order.

Unital ℓ-monoids

Unital ℓ-monoids

Definition
A *(totally distributive commutative) ℓ-monoid* is an algebra $\langle M; +, \vee, \wedge, 0\rangle$ such that:

M1. $\langle M; \vee, \wedge\rangle$ is a distributive lattice;

M2. $\langle M; +, 0\rangle$ is a commutative monoid;

M3. $+$ distributes over $\vee$ and $\wedge$.

A *unital ℓ-monoid* is an algebra $\langle M; +, \vee, \wedge, 0, 1, -1\rangle$ such that $\langle M; +, \vee, \wedge, 0\rangle$ is an ℓ-monoid and

U1. $-1 \leq 0 \leq 1$;

U2. $(-1) + 1 = 0$;

U3. For all $x \in M$, there exists $n \in \mathbb{N}$ such that $n(-1) \leq x \leq n1$.

Examples of unital ℓ-monoids

Example
$\mathbb{R}$ is a unital ℓ-monoid.

Example
$\mathbb{Z}$ is a unital ℓ-monoid.

Example
Given a compact space X with a partial order (e.g. $X = [a, b] \subseteq \mathbb{R}$),

$$C_{\leq}(X, \mathbb{R}) \coloneqq \{f : X \to \mathbb{R} \text{ continuous and monotone}\},$$
$$C_{\leq}(X, \mathbb{Z}) \coloneqq \{f : X \to \mathbb{Z} \text{ continuous and monotone}\}$$

are unital ℓ-monoids.

Example
$\mathbb{Z} \overrightarrow{\times} \mathbb{N} = \{k + n\varepsilon \mid k \in \mathbb{Z}, n \in \{0, 1, 2, \dots\}\}$ is a unital ℓ-monoid.

Example
$\{k - n\varepsilon \mid k \in \mathbb{Z}, n \in \{0, 1, 2, \dots\}\}$ is a unital ℓ-monoid.

Subdirectly irreducible ℓ-monoids

Theorem (Fuchs, unpublished; Merlier, 1971; Repnitzkii, 1984)
Every subdirectly irreducible ℓ-monoid is totally ordered.

Every subdirectly irreducible unital ℓ-monoid is totally ordered. Every unital ℓ-monoid is isomorphic to a subalgebra of a product of nontrivial totally ordered unital ℓ-monoids.

For a nontrivial totally ordered unital ℓ-monoid M, there exists a unique homomorphism $\varphi\colon M \to \mathbb{R}$. This homomorphism "kills the infinitesimals".

Representation

Let M be a unital ℓ-monoid. Set

$$Max(M) := \mathrm{hom}(M, \mathbb{R}).$$

$Max(M)$ can be endowed with a certain topology and a certain partial order.

Definition (Nachbin, 1965)
A *compact ordered space* is a compact Hausdorff space X with a partial order $\leq$ which is closed in $X \times X$.

$Max(M)$ is a compact ordered space.

We obtain a homomorphism

$$\begin{aligned} \mathrm{ev}\colon M &\longrightarrow C_{\leq}(Max(M), \mathbb{R}) \\ x &\longmapsto \mathrm{ev}_x : \varphi \mapsto \varphi(x). \end{aligned}$$

This homomorphism "kills the infinitesimals".

MMV-algebras

The unit interval of a unital ℓ-monoid

Given a unital ℓ-monoid M we set

$$\widetilde{\Gamma}(M) := \{x \in M \mid 0 \leq x \leq 1\}.$$

We endow $\widetilde{\Gamma}(M)$ with the following operations:

- $x \oplus y := (x + y) \wedge 1$;
- $x \odot y := (x + y - 1) \vee 0$;
- $\vee$ is defined by restriction;
- $\wedge$ is defined by restriction;
- $0 \in \widetilde{\Gamma}(M)$;
- $1 \in \widetilde{\Gamma}(M)$.

MMV-algebras

Definition

We call *MMV-algebra* (for *Monoidal MV-algebra*) an algebra $\langle A; \oplus, \odot, \vee, \wedge, 0, 1\rangle$ such that

A1. $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice;

A2. $\langle A; \oplus, 0\rangle$ and $\langle A; \odot, 1\rangle$ are commutative monoids;

A3. $x \oplus 1 = 1$ and $x \odot 0 = 0$;

A4. the operations $\oplus$ and $\odot$ distribute over $\vee$ and $\wedge$;

A5. $[(x \oplus y) \odot z] \oplus (x \odot y) = [(x \odot y) \oplus z] \odot (x \oplus y)$;

A6. $[(x \oplus y) \odot z] \oplus (x \odot y) = [(x \oplus z) \odot y] \oplus (x \odot z)$;

A7. $(x \oplus y) \odot z = \{[(x \oplus y) \odot z] \oplus (x \odot y)\} \wedge z$;

A8. $(x \odot y) \oplus z = \{[(x \oplus y) \odot z] \oplus (x \odot y)\} \vee z$.

In $[0, 1]$ (with $x \oplus y := \min\{x + y, 1\}$, $x \odot y := \max\{x + y - 1, 0\}$):

$[(x \oplus y) \odot z] \oplus (x \odot y) = \{[(x+y+z)\vee 1]\wedge 2\}-1 = [(x \odot y) \oplus z] \odot (x \oplus y)$.

MMV-algebras

MMV-algebras form a finitely axiomatised variety of finitary algebras!

Examples of MMV-algebras

Example
$[0,1]$ is an MMV-algebra.

Example
Given a unital ℓ-monoid M, $\langle \widetilde{\Gamma}(M); \oplus, \odot, \vee, \wedge, 0, 1 \rangle$ is an MMV-algebra.
(And every MMV-algebra is of this form.)

Example
Given a topological space X with a partial order (e.g. $X = [a,b] \subseteq \mathbb{R}$),

$$C_{\leq}(X,[0,1]) \coloneqq \{f \colon X \to [0,1] \text{ cont. and mon.}\} = \widetilde{\Gamma}(C_{\leq}(X,\mathbb{R}))$$

is an MMV-algebra.

Example
$\{0,1\} = \widetilde{\Gamma}(\mathbb{Z})$ is an MMV-algebra ($\oplus = \vee$ and $\odot = \wedge$).

Example
Every bounded distributive lattice L is an MMV-algebra, by setting $\oplus \coloneqq \vee$ and $\odot \coloneqq \wedge$. $L \simeq \widetilde{\Gamma}(C_{\leq}(Spec(L),\mathbb{Z}))$.

The equivalence

Main result: equivalence

Theorem (Main result)
The category of unital ℓ-monoids is equivalent to the category of MMV-algebras.

	MMV-algebras	*Unital ℓ-monoids*
Pro	Finitely axiomatised <u>variety</u> of finitary algebras.	Handy operations and axioms.
Con	Unwieldy operations and axioms.	Not first-order definable.

The unit interval functor $\widetilde{\Gamma}$

Given a unital ℓ-monoid M,

$$\langle \widetilde{\Gamma}(M); \oplus, \odot, \vee, \wedge, 0, 1 \rangle$$

is an MMV-algebra.

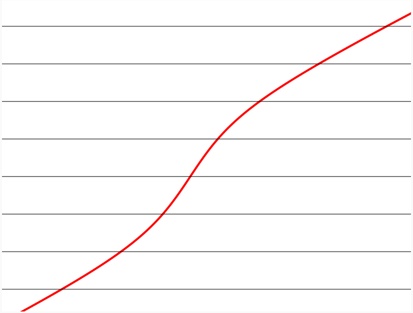
$\widetilde{\Gamma}$ defines a functor from unital ℓ-monoids to MMV-algebras.

A quasi-inverse of $\widetilde{\Gamma}$

We sketch the construction of a quasi-inverse for $\widetilde{\Gamma}$.

Idea
An element f of a unital ℓ-monoid M is determined by the function

$$\begin{aligned}\eta_M(f)\colon \mathbb{Z} &\to \widetilde{\Gamma}(M)\\ n &\mapsto [(f \vee n) \wedge (n+1)] - n.\end{aligned}$$



Good $\mathbb{Z}$-sequences

Definition
A *good $\mathbb{Z}$-sequence* in A is a function $\mathrm{x}\colon \mathbb{Z} \to A$ such that

1. definitely for $k \to -\infty$ we have $\mathrm{x}(k) = 1$;
2. definitely for $k \to +\infty$ we have $\mathrm{x}(k) = 0$;
3. for all $k \in \mathbb{Z}$, we have

$$\mathrm{x}(k) \oplus \mathrm{x}(k+1) = \mathrm{x}(k);$$
$$\mathrm{x}(k) \odot \mathrm{x}(k+1) = \mathrm{x}(k+1).$$

We set $\widetilde{\Xi}(A)$ as the set of good $\mathbb{Z}$-sequences in A.

$\widetilde{\Xi}(A)$ is a unital ℓ-monoid.

$\widetilde{\Xi}$ is a functor from MMV-algebras to unital ℓ-monoids.

Good $\mathbb{Z}$-sequences give quasi-inverse.

Proposition
$\widetilde{\Gamma}$ and $\widetilde{\Xi}$ are quasi-inverses.

Theorem (Main result)
The category of unital ℓ-monoids is equivalent to the category of MMV-algebras.

Classical Mundici's equivalence is a consequence.

The dual of compact ordered spaces

Dualities

Stone duality

Stone spaces (Comp. Hausd. 0-dimensional)

Boolean algebras $\vee, \wedge, 0, 1, \neg.$

Priestley duality

Priestley spaces (Stone space with a partial order + totally order-disconnectedness)

Bounded distributive lattices $\vee, \wedge, 0, 1.$

Dualities: above 0-dimensionality

Duality for compact Hausdorff spaces

Compact Hausdorff spaces	*MV-algebras + ...* $\oplus, \odot, \vee, \wedge, 0, 1, \neg, \ldots$.

Duality for compact ordered spaces

Compact ordered spaces (Compact Hausdorff space with a closed partial order)	*MMV-algebras + ...* $\oplus, \odot, \vee, \wedge, 0, 1, \ldots$.

Theorem
The category of compact ordered spaces is dually equivalent to a variety of infinitary algebras.

We present one such variety MMV_∞ using the signature $\{\oplus, \odot, \vee, \wedge\} \cup [0,1] \cup \{\delta\}$, where δ has countable arity.

The axioms for the variety

An algebra A belongs to MMV_∞ if

1. $\langle A; \oplus, \odot, \vee, \wedge, 0, 1\rangle$ is an MMV-algebra;
2. for $\alpha, \beta \in [0,1]$ s.t. $\alpha \leq \beta$ in $[0,1]$ we have $\alpha^A \leq \beta^A$;
3. for $\alpha, \beta, \gamma \in [0,1]$ s.t. $\alpha \oplus \beta = \gamma$ in $[0,1]$, we have $\alpha^A \oplus \beta^A = \gamma^A$;
4. for $\alpha, \beta, \gamma \in [0,1]$ s.t. $\alpha \odot \beta = \gamma$ in $[0,1]$, we have $\alpha^A \odot \beta^A = \gamma^A$;
5. $\delta(x, x, x, \dots) = x$;
6. $\delta(\tau_1(x,y), \tau_2(x,y), \tau_3(x,y), \dots) = x$;
7. $\forall n$: $\rho_n(x_1, \dots, x_n) \ominus \frac{1}{2^{n-1}} \leq \delta(x_1, x_2, x_3, \dots) \leq \rho_n(x_1, \dots, x_n) \oplus \frac{1}{2^{n-1}}$,

where

$$x \ominus \lambda \coloneqq x \odot (1 - \lambda) \quad (\text{for } \lambda \in [0,1]).$$

$$\tau_n(x,y) \coloneqq \left(y \vee \left(x \ominus \frac{1}{2^n}\right)\right) \wedge \left(x \oplus \frac{1}{2^n}\right).$$

$$\rho_1(x_1) \coloneqq x_1;$$

$$\rho_n(x_1, \dots, x_n) \coloneqq \tau_{n-1}(\rho_{n-1}(x_1, \dots, x_{n-1}), x_n) \quad (\text{for } n \geq 2).$$

Conclusions

Conclusions

Main result
The category of unital ℓ-monoids is equivalent to the category of MMV-algebras.

Future work

Future work

1. **Facts**:
   - a1. $\langle \mathbb{R}; +, \vee, \wedge, 0 \rangle$ does not generate the variety of ℓ-monoids.
   - a2. The variety generated by $\langle \mathbb{R}; +, \vee, \wedge, 0 \rangle$ is not finitely axiomatised.
   - a3. A countable equational axiomatisation for the variety generated by $\langle \mathbb{R}; +, \vee, \wedge, 0 \rangle$ is known.

   To do:
   - b1. Prove that $\langle [0,1]; \oplus, \odot, \vee, \wedge, 0, 1 \rangle$ does not generate the variety of MMV-algebras.
   - b2. Prove that the variety generated by $\langle [0,1]; \oplus, \odot, \vee, \wedge, 0, 1 \rangle$ is not finitely axiomatised.
   - b3. Provide a countable equational axiomatisation for the variety generated by $\langle [0,1]; \oplus, \odot, \vee, \wedge, 0, 1 \rangle$.

Future work

2. **Fact**:
   The class of $\{+, \vee, \wedge, 0\}$-subreducts of Abelian lattice-ordered groups is axiomatised by the equations defining ℓ-monoids together with the cancellation law:

$$x + z = y + z \Longrightarrow x = y.$$

   To do:
   Prove that the class of $\{\oplus, \odot, \vee, \wedge, 0, 1\}$-subreducts of MV-algebras is axiomatised by the equations defining MMV-algebras together with the single quasi-equation

$$\text{If } x \oplus z = y \oplus z \text{ and } x \odot z = y \odot z, \text{ then } x = y.$$

Thank you for your attention!