On the Axiomatisability of the Dual of Compact Ordered Spaces

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Compact ordered spaces

Compact ordered spaces: introduced by L. Nachbin in 1948 as an ordered version of compact Hausdorff spaces.

Definition ([Nachbin, 1948])

A *compact ordered space* is a compact space X with a partial order $\leq$ which is closed in $X \times X$.

CompOrd: category of compact ordered spaces and order-preserving continuous maps.

Examples

- $[0, 1]$ with Euclidean topology and $\leq$.
- Compact Hausdorff space with identity relation.
- Priestley space.
- Closed subspace of a power of $([0, 1], \leq)$.

Ordered-topological structures

$$\begin{aligned}\text{Compact ordered spaces} \ : \ \text{compact Hausdorff spaces} \\ = \ \text{Priestley spaces} \ : \ \text{Stone spaces.}\end{aligned}$$

Dualities with varieties

$$
\begin{array}{lll}
\text{Stone} & \xleftrightarrow{\text{op}} & \text{variety of finitary algebras [Stone, 1936]} \\
\text{Priestley} & \xleftrightarrow{\text{op}} & \text{variety of finitary algebras [Priestley, 1970]} \\
\text{CompHaus} & \xleftrightarrow{\text{op}} & \text{variety of (infinitary) algebras [Duskin, 1969]} \\
\text{CompOrd} & \xleftrightarrow{\text{op}} & ???
\end{array}
$$

Open question [Hofmann, Neves and Nora, 2018]

Is the category of compact ordered spaces dually equivalent to a variety of (possibly infinitary) algebras?

Known dualities for compact ordered spaces

CompOrd is known to be dually equivalent to the categories of:

1. stably compact frames;
2. strong proximity lattices.

However, neither of the two is a variety of algebras.

CompOrd$^{\text{op}}$ is a quasivariety

As observed by [Hofmann, Neves and Nora, 2018], CompOrd is dually equivalent to a *quasi*variety of (possibly infinitary) algebras: this follows from

1. results of [Nachbin, 1965], and
2. categorical characterisations of quasivarieties.

Function symbols of arity a cardinal κ: order-preserving continuous functions $[0,1]^\kappa \to [0,1]$. They have obvious interpretations on $[0,1]$.

Full, faithful, essentially surjective *contravariant* functor

$$\begin{aligned} \text{CompOrd} &\xrightarrow[\sim]{\text{op}} \mathbb{SP}([0,1]) \\ X &\longmapsto \{f \colon X \to [0,1] \mid f \text{ is order-pres. and cont.}\}. \end{aligned}$$

Results known from the literature

Theorem ([Hofmann, Neves and Nora, 2018])

CompOrd *is dually equivalent to an* $\aleph_1$*-ary quasivariety.*

($\aleph_1$-ary *quasi*variety: function symbols of at most countable arity, implications with at most countably many premises.)

$\mathsf{CompOrd}^{\mathrm{op}}$ is a variety

Main result:
The category of compact ordered spaces is dually equivalent to a variety of algebras, with operations of at most countable arity.

Negative results

Theorem

CompOrd *is not dually equivalent to any variety of finitary algebras.*
In fact, CompOrd *is not dually equivalent to*

1. *any finitely accessible category (the same holds for any full subcategory of* CompOrd *which strictly contains* Priestley*);*
2. *any first-order definable class of structures (no faithful functor* $\mathsf{CompOrd}^{\mathrm{op}} \to \mathsf{Set}$ *preserves directed colimits) [Lieberman, Rosický and Vasey, 2019];*
3. *any class of finitary algebras closed under products and subalgebras.*

Proof of main result

Known: $\mathsf{CompOrd}^{\mathsf{op}}$ is (equivalent to) a quasivariety ($\mathbb{SP}([0,1])$). Is this quasivariety also a variety?

Known: for a quasivariety A,

$$\begin{gathered}\text{A is a variety}\\ \Updownarrow\\ \text{A is closed under homomorphic images}\\ \Updownarrow\\ \text{congruences are kernels}\\ \Updownarrow\\ \text{(internal) equivalence relations are effective.}\end{gathered}$$

We shall prove that equivalence relations in $\mathsf{CompOrd}^{\mathsf{op}}$ are effective.

Effectiveness of equivalence relations

Every equivalence relation in CompHaus$^{\mathrm{op}}$ is effective: proof of 12 lines in [Barr and Wells, 1985].

However, this proof does not work for CompOrd$^{\mathrm{op}}$.
$\rightarrow$ The addition of the order complicates the proof.

Binary relations and their duals

Binary relation on X in CompOrd$^{\mathrm{op}}$

$\updownarrow$

Equiv. class of monomorphism $R \hookrightarrow X \times X$ in CompOrd$^{\mathrm{op}}$

$\updownarrow$

Equiv. class of epimorphism $X + X \twoheadrightarrow R$ in CompOrd

$\updownarrow$

Closed preorder on $X + X$ which extends $\leq_{X+X}$

We characterise the notions of reflexivity, symmetry, transitivity and effectiveness at this last level.

Effectiveness

Theorem

Every equivalence relation in $\mathsf{CompOrd}^{\mathrm{op}}$ *is effective.*

Corollary (Main result)

The category of compact ordered spaces is dually equivalent to a variety of algebras, with operations of at most countable arity.

The variety

Algebraic theory (in the sense of Lawvere-Linton) of CompOrd$^{\mathrm{op}}$:

Objects: (possibly infinite) powers of $[0,1]$.

Morphisms: order-preserving continuous functions.

A variety dually equivalent to CompOrd is

$$\mathbb{SP}([0,1]),$$

where the function symbols of arity κ are the order-preserving continuous functions $[0,1]^\kappa \to [0,1]$. (Any such function depends on at most countably many coordinates.)

Any algebra in $\mathbb{SP}([0,1])$ is isomorphic to

$$\{f \colon X \to [0,1] \mid f \text{ is order-pres. and cont.}\}$$

for a unique compact ordered space X.

Barr-exactness

$\mathsf{CompOrd}^{\mathrm{op}}$ is Barr-exact. Then, so are the categories of strong proximity lattices and of stably compact frames.

Finite equational axiomatisation

Does there exist a manageable axiomatisation of CompOrd$^{\mathrm{op}}$?

CompOrd$^{\mathrm{op}}$ admits a finite equational axiomatisation, i.e. one which uses only finitely many function symbols and finitely many equational axioms.

Primitive operations

Primitive operations: $\oplus$, $\odot$, $\vee$, $\wedge$, 0, 1, h, j, λ (arities: 2, 2, 2, 2, 0, 0, 1, 1, ω).

$$
\begin{aligned}
x \oplus y &\coloneqq \min\{x + y, 1\},\\
x \odot y &\coloneqq \max\{x + y - 1, 0\},\\
x \vee y &\coloneqq \max\{x, y\},\\
x \wedge y &\coloneqq \min\{x, y\},\\
0 &\coloneqq 0,\\
1 &\coloneqq 1,\\
\mathrm{h}(x) &\coloneqq \frac{x}{2},\\
\mathrm{j}(x) &\coloneqq \frac{1}{2} + \frac{x}{2}.
\end{aligned}
$$

The operations generated by $\oplus$, $\odot$, $\vee$, $\wedge$, 0, 1, h, j approximate any order-preserving continuous function $[0, 1]^{\kappa} \to [0, 1]$.

The limit-like operation

$$\lambda(x_1, x_2, x_3, \dots) \coloneqq \lim_{n\to\infty} \mu_n(x_1, \dots, x_n),$$

where μ_n is defined inductively:

$$\begin{aligned}\mu_1(x_1) &\coloneqq x_1,\\ \mu_n(x_1,\dots,x_n) &\coloneqq \max\left\{\min\left\{x_n, \mu_{n-1}(x_1,\dots,x_{n-1}) + \frac{1}{2^n}\right\},\right.\\ &\qquad\qquad\left.\mu_{n-1}(x_1,\dots,x_{n-1}) - \frac{1}{2^{n-1}}\right\}.\end{aligned}$$

For 'sufficiently many' sequences $(x_1, x_2, x_3, \dots)$, we have

$$\lambda(x_1, x_2, x_3, \dots) = \lim_{n\to\infty} x_n.$$

Recap

Negative results: CompOrd is not dually equivalent to

- any finitely accessible category;
- any first-order definable class of structures;
- any class of finitary algebras closed under products and subalgebras.

In particular, CompOrd is not dually equivalent to any variety of finitary algebras.

Positive results: CompOrd is dually equivalent to a variety of algebras described by

- *finitely* many function symbols of at most *countable* arity, and
- *finitely* many equational axioms.

Generalisation of Mundici's theorem

En passant, in the search for a reasonable set of axioms for $\oplus$, $\odot$, $\vee$, $\wedge$, 0, 1, a generalisation of a theorem by D. Mundici was obtained.

Mundici's theorem [Mundici, 1986]: the categories of unital Abelian ℓ-groups and of MV-algebras are equivalent.

Generalisation: The category of unital commutative distributive *ℓ-monoids* is equivalent to the category of MV-monoidal algebras.

Lattice-ordered monoids

Definition

Unital commutative distributive ℓ-monoid: $\langle M; +, \vee, \wedge, 0, 1, -1\rangle$ s.t.

1. $\langle M; \vee, \wedge\rangle$ is a distributive lattice.
2. $\langle M; +, 0\rangle$ is a commutative monoid.
3. The operation $+$ distributes over $\vee$ and $\wedge$.
4. $-1 \leq 0 \leq 1$.
5. $-1 + 1 = 0$.
6. $\forall x \in M, \exists n \in \mathbb{N}$ s.t. $n(-1) \leq x \leq n1$.

Example

For X a compact ordered space,

$$\{f \colon X \to \mathbb{R} \mid f \text{ is order-preserving and continuous}\}.$$

Unit interval functor

Given a unital commutative distributive ℓ-monoid M, one equips the set

$$\Gamma(M) \coloneqq \{x \in M \mid 0 \leq x \leq 1\}$$

with the operations $\vee$, $\wedge$, 0, and 1 by restriction, and

$$x \oplus y \coloneqq (x + y) \wedge 1,$$
$$x \odot y \coloneqq (x + y - 1) \vee 0.$$

Example

$\Gamma(\{\text{order-pres. cont. } X \to \mathbb{R}\}) = \{\text{order-pres. cont. } X \to [0, 1]\}$.

$\langle \Gamma(M); \oplus, \odot, \vee, \wedge, 0, 1 \rangle$ completely captures M.

MV-monoidal algebras

Definition

MV-monoidal algebra: $\langle A; \oplus, \odot, \vee, \wedge, 0, 1\rangle$ s.t.

1. $\langle A; \vee, \wedge\rangle$ is a distributive lattice.
2. $\langle A; \oplus, 0\rangle$ and $\langle A; \odot, 1\rangle$ are commutative monoids.
3. Both the operations $\oplus$ and $\odot$ distribute over both $\vee$ and $\wedge$.
4. $(x \oplus y) \odot ((x \odot y) \oplus z) = (x \odot (y \oplus z)) \oplus (y \odot z)$.
5. $(x \odot y) \oplus z = ((x \oplus y) \odot ((x \odot y) \oplus z)) \vee z$.
6. $(x \oplus y) \odot z = ((x \odot y) \oplus ((x \oplus y) \odot z)) \wedge z$.

Theorem

The categories of unital commutative distributive ℓ-monoids and of MV-monoidal algebras (with homomorphisms) are equivalent.

This gives us a reasonable set of axioms for $\oplus, \odot, \vee, \wedge, 0, 1$.

Vietoris functor

We have a 'Vietoris' endofunctor $V \colon \mathsf{CompOrd} \to \mathsf{CompOrd}$ (see [Schalk, 1993, Hofmann and Nora, 2018]).

Theorem ([Hofmann, Neves and Nora, 2018])

The category of coalgebras for $V \colon \mathsf{CompOrd} \to \mathsf{CompOrd}$ is dually equivalent to an $\aleph_1$-ary quasivariety.

Theorem

The category of coalgebras for $V \colon \mathsf{CompOrd} \to \mathsf{CompOrd}$ is dually equivalent to a variety, with operations of at most countable arity.

Thank you for your attention.

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