# The dual of compact ordered spaces is a variety

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### Definition

A *compact pospace* $(X, \leqslant)$ is a compact topological space $X$, endowed with a partial order $\leqslant$, so that the set

$$\{(x, y) \in X \times X \mid x \leqslant y\}$$

is closed in $X \times X$ with respect to the product topology.

Example. $[0, 1]$.

# Representation of compact pospaces

For any set $I$, endow $[0,1]^I$ with the product topology and product order:

$$\text{for } p, q \in [0,1]^I, p \leqslant q \Leftrightarrow \forall\, i \in I \;\; p_i \leqslant q_i.$$

Compact pospaces = closed subspaces of $[0,1]^I$ (for some set $I$).

PosComp: category of compact pospaces with monotone continuous maps.

[Monotonicity: $x \leqslant y \Rightarrow f(x) \leqslant f(y)$.]

In *Generating the algebraic theory of* $C(X)$*: the case of partially ordered compact spaces* (2018), Hofmann, Neves and Nora proved the following.

PosComp$^{\mathrm{op}}$ *is equivalent to a quasi-variety.*

That means, there is a set $\tau$ of function symbols and a set of quasi-equations

$$\left(\bigwedge_i \sigma_i = \rho_i\right) \Rightarrow \gamma = \eta$$

such that PosComp$^{\mathrm{op}}$ is equivalent to the category of $\tau$-algebras satisfying the given quasi-equations.
[Such quasi-variety is not finitary: one of the primitive function symbols has countably infinite arity.]

In the same paper, they left as open the following question.

### Question

Is $\mathrm{PosComp}^{\mathrm{op}}$ equivalent to a variety?

That means, is there a set $\tau$ of function symbols and a set of equations

$$\gamma = \eta$$

such that $\mathrm{PosComp}^{\mathrm{op}}$ is equivalent to the category of $\tau$-algebras satisfying the given equations?

**Theorem** (Main result)
*The dual of* PosComp *is equivalent to a variety.*

Rest of the talk: sketch of the proof.

We provide

- a set of finitary function symbols, and a single function symbol $\delta$ of countably infinite arity,
- a set of equations,

such that $\mathrm{PosComp}^{\mathrm{op}}$ is equivalent to the category of algebras satisfying the given equations.

## Strategy

1. **Dual adjunction** between PosComp and a finitary variety $\mathcal{V}$, to be defined. Contravariant functors:
   $C\colon \text{PosComp} \to \mathcal{V}$,
   $Max\colon \mathcal{V} \to \text{PosComp}$.
2. **Fixed objects**:
   - in PosComp: every object.
   - in $\mathcal{V}$: archimedean Cauchy complete $\mathcal{V}$-algebras.

   $\Rightarrow$ duality between PosComp and archimedean Cauchy complete $\mathcal{V}$-algebras.
3. **The variety** $\mathcal{V}_\delta$.

   Subcategory of archimedean Cauchy complete $\mathcal{V}$-algebras

   $$\cong$$

   (infinitary) variety $\mathcal{V}_\delta$.

# 1. Dual adjunction

Let $X$ be a compact pospace.

$$C(X) := \{f : X \to [0,1] \mid f \text{ is continuous and monotone}\}.$$

Which internal operations can we define on $C(X)$?

Example: for $f, g \in C(X)$, the function $f \vee g$ belongs to $C(X)$, where $f \vee g$ is the pointwise application to $f$ and $g$ of the binary supremum $\vee : [0,1]^2 \to [0,1]$.

### Proposition

*Let $I$ be a set, and let $\tau\colon [0,1]^I \to [0,1]$ be a continuous monotone function. Then, for $X$ a compact pospace, $\mathrm{C}(X)$ is closed under $\tau$ (pointwise applied), i.e.,*

$$\forall\ (f_i)_{i\in I} \subseteq \mathrm{C}(X) \text{ we have } x \mapsto \tau((f_i(x))_{i\in I}) \in \mathrm{C}(X).$$

Idea: the terms of the variety $\mathcal{V}_\delta$ dual to PosComp "are" the monotone continuous functions $\tau\colon [0,1]^I \to [0,1]$.

We shall obtain every monotone continuous function $[0,1]^I \to [0,1]$ via composition of the following ones.

1. $a \vee b := \max\{a, b\}$.
2. $a \wedge b := \min\{a, b\}$.
3. $a \oplus b := \min\{a + b, 1\}$.
4. $a \odot b := \max\{a + b - 1, 0\}$.
5. Each constant $\lambda \in [0,1]$.
6. A function $\delta\colon [0,1]^{\mathbb{N}} \to [0,1]$, to be defined.

## The variety $\mathcal{V}$.

A $\mathcal{V}$-algebra is a set $A$, endowed with internal operations $\{\vee, \wedge, \oplus, \odot\} \cup \{\lambda \mid \lambda \in [0,1]\}$ such that:

1. $(A, \vee, \wedge, 0, 1)$ is a distributive bounded lattice;
2. $(A, \oplus, 0)$ is commutative monoid with absorbing element 1;
3. $(A, \odot, 1)$ is commutative monoid with absorbing element 0;
4. $\oplus$ and $\odot$ distributes over $\vee$ and $\wedge$;
5. $(a \oplus b) \odot c \leqslant a \oplus (b \odot c)$;
6. For each $\lambda \in [0,1]$, $a \leqslant (a \odot (1-\lambda)) \oplus \lambda$;
7. For each $\lambda \in [0,1]$, $a \geqslant (a \oplus \lambda) \odot (1-\lambda)$;
8. For every $n, m \in \{0, 1, 2, \dots\}$, we have the axiom
   $$a \wedge (b \oplus \underbrace{(c \odot \lambda) \oplus \cdots \oplus (c \odot \lambda)}_{n \text{ times}}) \leqslant$$
   $$(a \odot \underbrace{(c \oplus \lambda) \odot \cdots \odot (c \oplus \lambda)}_{m \text{ times}}) \vee b;$$
9. $\{\vee, \wedge, \oplus, \odot\}$ operate on the constant symbols $\lambda \in [0,1]$ as their intended interpretation in $[0,1]$ do.

$[0,1]$ is a $\mathcal{V}$-algebra (and a compact pospace).

For $X$ compact pospace,

$$C(X) = \{f\colon X \to [0,1] \mid f \text{ is continuous and monotone}\}$$

is a $\mathcal{V}$-algebra, with pointwise applied operations.

$C\colon \mathrm{PosComp} \to \mathcal{V}$.

For $A \in \mathcal{V}$, set

$$Max(A) := \hom_{\mathcal{V}}(A, [0,1]).$$

Topology on $Max(A)$: the smallest one to which belong, for every $a \in A$ and $O$ open subset of $[0,1]$, $\{x \in Max(A) \mid x(a) \in O\}$.

Order on $Max(A)$: $x \leqslant y$ if, and only if, for all $a \in A$, $x(a) \leqslant y(a)$.

### Theorem

*For $A \in \mathcal{V}$, $Max(A)$ is a compact pospace.*

Proof (L. Reggio).

Sketch: $Max(A) = \hom_{\mathcal{V}}(A, [0,1]) \subseteq [0,1]^A$. Given the definition of $Max(A)$, and the fact that $[0,1]^A$ is a compact pospace with respect to the product topology and the pointwise order, it is enough to prove that $Max(A)$ is a closed subset of $[0,1]^A$. The idea is that $Max(A)$ is closed because it is defined via equations, that express the preservation of the primitive function symbols. $\square$

Contravariant functors:
$C\colon \mathrm{PosComp} \to \mathcal{V}$,
$Max\colon \mathcal{V} \to \mathrm{PosComp}$.

They are (dually) adjoint.

For $X \in \mathrm{PosComp}$, the unit is

$$\begin{aligned} ev_X\colon X &\to Max(C(X)) \\ x &\mapsto (ev_x\colon C(X) \to [0,1]; a \mapsto a(x)). \end{aligned}$$

For $A \in \mathcal{V}$, the unit is

$$\begin{aligned} ev_A\colon A &\to C(Max(A)) \\ a &\mapsto (ev_a\colon Max(A) \to [0,1]; x \mapsto x(a)). \end{aligned}$$

# 2. Fixed objects

## 2a. Fixed objects in PosComp

**Theorem**

*For every $X$ compact pospace, the unit $ev_X$ is an isomorphism.*

## 2b. Fixed objects in $\mathcal{V}$

Let $A$ be a $\mathcal{V}$-algebra. For $a, b \in \mathcal{V}$, set

$$\mathrm{d}(a, b)$$

as the maximum between

$$\mathrm{d}^{\uparrow}(a, b) := \inf\{\lambda \in [0,1] \mid b \leqslant a \oplus \lambda\},$$

and

$$\mathrm{d}^{\uparrow}(b, a) = \inf\{\lambda \in [0,1] \mid a \leqslant b \oplus \lambda\}.$$

When $A \subseteq [0,1]^X$, d coincides with the sup metric.

1. $\mathrm{d}(a,b) \geqslant 0$.
2. $\mathrm{d}(a,b) = \mathrm{d}(b,a)$.
3. $\mathrm{d}(a,c) \leqslant \mathrm{d}(a,b) + \mathrm{d}(b,c)$.
4. $\mathrm{d}(a,a) = 0$.
5. $\mathrm{d}(a,b) = 0 \Rightarrow a = b$ ???

## Definition

$A \in \mathcal{V}$ is called *archimedean* if, for all $a, b \in A$,

$$\mathrm{d}(a, b) = 0 \Rightarrow a = b.$$

## Theorem

*Let $A \in \mathcal{V}$. The following conditions are equivalent.*

1. *$A$ is archimedean.*
2. *For every $x, y \in A$ with $x \neq y$, there exists a $\mathcal{V}$-morphism $\varphi \colon A \to [0,1]$ such that $\varphi(x) \neq \varphi(y)$.*
3. *There exists a set $X$ such that $A$ is a $\mathcal{V}$-subalgebra of $[0,1]^X$.*
4. *The unit $ev_A \colon A \to C(Max(A))$ is injective.*

To prove [1.⇒2.] we make use of the Subdirect Representation Theorem, which applies since $\mathcal{V}$ has only finitary terms.

Which property of $A$ is missing in order to have an isomorphism $A \cong C(Max(A))$?
Cauchy completeness.

## Definition

Let $A \in \mathcal{V}$.

- $(a_n)_{n\in\mathbb{N}} \subseteq A$ is *Cauchy* if
$$\forall \varepsilon > 0, \exists k \in \mathbb{N}\colon \forall n, m \geqslant k, \mathrm{d}(a_n, a_m) < \varepsilon.$$
- $(a_n)_{n\in\mathbb{N}} \subseteq A$ *converges to* $a \in A$ if
$$\forall \varepsilon > 0, \exists n \in \mathbb{N}\colon \forall m \geqslant n, \mathrm{d}(a_m, a) < \varepsilon.$$
- $(a_n)_{n\in\mathbb{N}} \subseteq A$ *converges* if there is $a \in A$ such that $(a_n)_{n\in\mathbb{N}}$ converges to $a$.
- $A$ is *Cauchy complete* if every Cauchy sequence in $A$ converges.

### Theorem

*A $\mathcal{V}$-algebra $A$ is Cauchy complete if, and only if, the unit $ev_A \colon A \to \mathsf{C}(Max(A))$ is surjective.*

### Theorem

*Let $A \in \mathcal{V}$. The following conditions are equivalent.*

1. *The unit $ev_A : A \to \mathsf{C}(Max(A))$ is an isomorphism.*
2. *There exists $\mathsf{X}$ compact pospace such that $A$ and $\mathsf{C}(X)$ are isomorphic $\mathcal{V}$-algebras.*
3. *$A$ is archimedean and Cauchy complete.*

# 3. The variety $\mathcal{V}_\delta$

Up to now: dual adjunction between PosComp and $\mathcal{V}$, which restricts to a duality between PosComp and the archimedean Cauchy complete $\mathcal{V}$-algebras.

Final goal:

$$\text{Subcategory of archimedean Cauchy complete } \mathcal{V}\text{-algebras}$$

$$\cong$$

$$\text{(infinitary) variety } \mathcal{V}_\delta.$$

### Idea

Add an operation $\delta$ of countably infinite arity to the set of operations of $\mathcal{V}$, together with some new axioms, so that

1. any model is an archimedean $\mathcal{V}$-algebra,
2. $\delta$ calculates the limit of *enough* Cauchy sequences.

## Definition

Let $A \in \mathcal{V}$. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ is called *HNN-Cauchy* if, for every $n \in \mathbb{N}$,

$$x_n \leqslant x_{n+1} \leqslant x_n \oplus \frac{1}{2^n}.$$

$\square$

Every HNN-Cauchy sequence is a Cauchy sequence.

## Proposition

*For $A \in \mathcal{V}$, $A$ is Cauchy complete if, and only if, every HNN-Cauchy sequence in $A$ converges.*

The intended interpretation of $\delta$ is of the following form.

$$\delta(x_0, x_1, x_2, \dots) = \lim_{n\to\infty} \rho_n(x_0, \dots, x_n),$$

where

1. if $(x_n)_{n\in\mathbb{N}}$ is an HNN-Cauchy sequence, then, for all $n \in \mathbb{N}$, $\rho_n(x_0, \dots, x_n) = x_n$;
2. $(\rho_n(x_0, \dots, x_n))_{n\in\mathbb{N}}$ is an HNN-Cauchy sequence.

If we find a sequence of terms $(\rho_n)_{n\in\mathbb{N}}$ in the language of $\mathcal{V}$ that satisfies (1) and (2), then $\delta$ is well-defined on any archimedean Cauchy complete $\mathcal{V}$-algebra, and it calculates the limit of HNN-Cauchy sequences.

$$\rho_0(x_0) = x_0;$$

$$\rho_{n+1}(x_0, \ldots, x_{n+1}) := (x_0 \vee \cdots \vee x_{n+1}) \wedge \left(\rho_n(x_0, \ldots, x_n) \oplus \frac{1}{2^n}\right).$$

## Proposition

*Let $A \in \mathcal{V}$.*

1. *If $(x_n)_{n\in\mathbb{N}} \subseteq A$ is an HNN-Cauchy sequence, then, for all $n \in \mathbb{N}$,*

$$\rho_n(x_0, \ldots, x_n) = x_n.$$

2. *For any sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$, $(\rho_n(x_0, \ldots, x_n))_{n\in\mathbb{N}}$ is an HNN-Cauchy sequence.*

(HNN-Cauchy: $x_n \leqslant x_{n+1} \leqslant x_n \oplus \frac{1}{2^n}$.)

Let $A \in \mathcal{V}$ be archimedean and Cauchy complete.

$$\delta(x_0, x_1, x_2, \dots) := \lim_{n \to \infty} \rho_n(x_0, \dots, x_n).$$

$\delta$ calculates the limit of HNN-Cauchy sequences.

Which equational axioms capture the behaviour of $\delta$?

## The variety $\mathcal{V}_\delta$

**Definition**

The variety $\mathcal{V}_\delta$ is the (infinitary) variety obtained from the variety $\mathcal{V}$ by adding an operation $\delta$ of countably infinite arity, together with the following additional axioms.

1. $\delta(x, x, x, \dots) = x$.
2. $\delta(x_0, x_1, x_2, \dots) \leqslant \delta(x_0 \vee y_0, x_1 \vee y_1, x_2 \vee y_2, \dots)$.
3. $\delta\left(x \ominus \frac{1}{2^0}, x \ominus \frac{1}{2^1}, x \ominus \frac{1}{2^2}, \dots\right) = x$.
4. (For all $n \in \mathbb{N}$)

$$\rho_n(x_0, \dots, x_n) \leqslant \delta(x_0, x_1, x_2, \dots) \leqslant \rho_n(x_0, \dots, x_n) \oplus \frac{1}{2^{n-1}}.$$

$\square$

Notation: $x \ominus \lambda := x \odot (1 - \lambda)$.

1. $\delta(x, x, x, \dots) = x$.
2. $\delta(x_0, x_1, x_2, \dots) \leqslant \delta(x_0 \vee y_0, x_1 \vee y_1, x_2 \vee y_2, \dots)$.
3. $\delta\left(x \ominus \frac{1}{2^0}, x \ominus \frac{1}{2^1}, x \ominus \frac{1}{2^2}, \dots\right) = x$.

## Proposition

*Every $A \in \mathcal{V}_\delta$ is archimedean.*

Proof.
Let $\mathrm{d}(x, y) = 0$. Goal: $x = y$. For all $\lambda \in (0, 1]$, we have $y \ominus \lambda \leqslant x$. Therefore

$$y \overset{(3)}{=} \delta\left(y \ominus \frac{1}{2^0}, y \ominus \frac{1}{2^1}, y \ominus \frac{1}{2^2}, \dots\right) \overset{(2)}{\leqslant} \delta(x, x, x, \dots) \overset{(1)}{=} x.$$

Hence, $y \leqslant x$. Analogously, $x \leqslant y$. Thus, $x = y$. □

4. $\rho_n(x_0, \ldots, x_n) \leqslant \delta(x_0, x_1, x_2, \ldots) \leqslant \rho_n(x_0, \ldots, x_n) \oplus \frac{1}{2^{n-1}}$.

## Proposition

*Every $A \in \mathcal{V}_\delta$ is Cauchy complete.*

Proof.
Let $(x_n)_{n\in\mathbb{N}}$ be an HNN-Cauchy sequence. Goal: it converges. Since $(x_n)_{n\in\mathbb{N}}$ is HNN-Cauchy, $\rho_n(x_0, \ldots, x_n) = x_n$. Hence $x_n \leqslant \delta(x_0, x_1, x_2, \ldots) \leqslant x_n \oplus \frac{1}{2^{n-1}}$. THus, $\mathrm{d}(x_n, \delta(x_0, x_1, x_2, \ldots)) \leqslant \frac{1}{2^{n-1}}$. Therefore, $(x_n)_{n\in\mathbb{N}}$ converges to $\delta(x_0, x_1, x_2, \ldots)$. □

Let $U\colon \mathcal{V}_\delta \to \mathcal{V}$ be the forgetful functor.

## Theorem

1. *$U$ is faithful.*
2. *$U$ is full.*
3. *$U$ is injective on objects.*
4. *The image of $U$ is the class of archimedean Cauchy complete $\mathcal{V}$-algebras.*

## Corollary

*The variety $\mathcal{V}_\delta$ is isomorphic to the full subcategory of $\mathcal{V}$ given by the archimedean Cauchy complete $\mathcal{V}$-algebras.*

# Summary

1. **Dual adjunction** between PosComp and a finitary variety $\mathcal{V}$.
2. **Fixed objects**:
   - in PosComp: every object.
   - in $\mathcal{V}$: archimedean Cauchy complete $\mathcal{V}$-algebras.

   $\Rightarrow$ duality between PosComp and archimedean Cauchy complete $\mathcal{V}$-algebras.
3. **The variety $\mathcal{V}_\delta$.**

   Subcategory of archimedean Cauchy complete $\mathcal{V}$-algebras

   $$\cong$$

   (infinitary) variety $\mathcal{V}_\delta$.

# Conclusion

**Theorem** (Main result)

*The dual of* PosComp *is equivalent to a variety.*

Thank you for your attention.