Equivalence à la Mundici for lattice-ordered monoids

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Motivation

Dualities

Stone duality

Stone spaces

(Comp. Hausd. 0-dimensional)

Boolean algebras

$C(X, \{0,1\}) :=$

$\{f\colon X \to \{0,1\} \text{ continuous}\}$

$\vee, \wedge, 0, 1, \neg.$

Priestley duality

Priestley spaces

(Stone space with a partial order

+ totally order-disconnectedness)

Distributive lattices

$C_{\leq}(X, \{0,1\}) :=$

$\{f\colon X \to \{0,1\} \text{ cont. and monot.}\}$

$\vee, \wedge, 0, 1.$

Dualities: above 0-dimensionality

Duality for compact Hausdorff spaces

Compact Hausdorff spaces

MV-algebras + ...
$C(X, [0,1]) :=$
$\{f : X \to [0,1] \text{ continuous}\}$
$\oplus, \odot, \vee, \wedge, 0, 1, \neg, \ldots$.

Duality for compact ordered spaces

Compact ordered spaces
(Comp. Hausd. space X with a partial order $\leq$, which is closed in $X \times X$)

???
$C_{\leq}(X, [0,1]) :=$
$\{f : X \to [0,1] \text{ cont. and monot.}\}$
$\oplus, \odot, \vee, \wedge, 0, 1, \ldots$.

Motivation

Recap
For some reason of some interest ($=$ generalisation of Priestley duality), one wants to study algebras of $[0,1]$-valued continuous monotone functions.

Fact
It is easier to study algebras of $\mathbb{R}$-valued continuous monotone functions.

Aim
Make a bridge between algebras of $\mathbb{R}$-valued and $[0,1]$-valued continuous monotone functions.

The two categories

Unital ℓ-monoids

Definition
An *ℓ-monoid* is a set M, endowed with operations $+, \vee, \wedge, 0$ such that:

M1. $\langle M; \vee, \wedge \rangle$ is a distributive lattice;

M2. $\langle M; +, 0 \rangle$ is a commutative monoid;

M3. $+$ distributes over $\vee$ and $\wedge$:

A *unital ℓ-monoid* is an ℓ-monoid M with two distinguished elements $1, -1 \in M$ such that:

U1. $-1 \leq 0 \leq 1$;

U2. $(-1) + 1 = 0$;

U3. For all $x \in M$, there exists $n \in \mathbb{N}$ such that $n(-1) \leq x \leq n1$.

Unital ℓ-monoids

Example
$\mathbb{R}$ is a unital ℓ-monoid.

Example
Given a compact ordered space X (e.g. $X = [0, 1]$),

$$C_{\leq}(X, \mathbb{R}) \coloneqq \{f : X \to \mathbb{R} \mid f \text{ is continuous and monotone}\}$$

is a unital ℓ-monoid.

A *morphism of unital ℓ-monoids* is a map that preserves $+, \vee, \wedge, 0, 1, -1$.

The unit interval of a unital ℓ-monoid

Idea
A unital ℓ-monoid M is determined by

$$\widetilde{\Gamma}(M) \coloneqq \{x \in M \mid 0 \leq x \leq 1\},$$

endowed with the following operations:

- $x \oplus y \coloneqq (x + y) \wedge 1$;
- $x \odot y \coloneqq (x + y - 1) \vee 0$;
- $\vee$ is defined by restriction;
- $\wedge$ is defined by restriction;
- 0 belongs to $\widetilde{\Gamma}(M)$;
- 1 belongs to $\widetilde{\Gamma}(M)$.

Which axioms are satisfied by $\widetilde{\Gamma}(M)$?

MMV-algebras

Definition

We call *MMV-algebra* (for *Monoidal MV-algebra*) an algebra $\mathbf{A} = \langle A; \oplus, \odot, \vee, \wedge, 0, 1 \rangle$ such that

(A1) $\langle A; \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice;

(A2) $\langle A; \oplus, 0 \rangle$ and $\langle A; \odot, 1 \rangle$ are commutative monoids;

(A3) $x \oplus 1 = 1$ and $x \odot 0 = 0$;

(A5) the operations $\oplus$ and $\odot$ distribute over $\vee$ and $\wedge$;

(A6) $[(x \oplus y) \odot z] \oplus (x \odot y) = [(x \odot y) \oplus z] \odot (x \oplus y)$;

(A7) $[(x \oplus y) \odot z] \oplus (x \odot y) = [(x \odot z) \oplus y] \odot (x \oplus z)$;

(A8) $(x \oplus y) \odot z = \{[(x \oplus y) \odot z] \oplus (x \odot y)\} \wedge z$;

(A9) $(x \odot y) \oplus z = \{[(x \odot y) \oplus z] \odot (x \oplus y)\} \vee z$.

MMV-algebras form a finitely based variety of finitary algebras!

MMV-algebras

Example
$[0,1]$ is an MMV-algebra.

Example
Given a compact ordered space X (e.g. $X = [0,1]$),

$$C_{\leq}(X,[0,1]) \coloneqq \{f \colon X \to [0,1] \mid f \text{ is continuous and monotone}\}$$

is an MMV-algebra.

Example
Every distributive lattice is an MMV-algebra, by setting $\oplus \coloneqq \vee$ and $\odot \coloneqq \wedge$.

A *morphism of MMV-algebras* is a map that preserves $\oplus, \odot, \vee, \wedge, 0, 1$.

The equivalence

Main result: equivalence

Theorem (Main result)
The category of unital ℓ-monoids is equivalent to the category of MMV-algebras.

	MMV-algebras	*Unital ℓ-monoids*
Pro	Finitely based variety of finitary algebras.	Handy operations and axioms.
Con	Unwieldy operations and axioms.	Not first-order definable.

The unit interval functor $\widetilde{\Gamma}$

Given a unital ℓ-monoid M,

$$\widetilde{\Gamma}(M) := \{x \in M \mid 0 \leq x \leq 1\}$$

is an MMV-algebra, where

- $x \oplus y := (x + y) \wedge 1$;
- $x \odot y := (x + y - 1) \vee 0$;
- $\vee$ is defined by restriction;
- $\wedge$ is defined by restriction;
- 0 belongs to $\widetilde{\Gamma}(M)$;
- 1 belongs to $\widetilde{\Gamma}(M)$.

$\widetilde{\Gamma}$ defines a functor from the category of unital ℓ-monoids to the category of MMV-algebras.

Goal: to construct a quasi-inverse of $\widetilde{\Gamma}$.

Idea to construct a quasi-inverse of $\widetilde{\Gamma}$

Idea
An element f of a unital ℓ-monoid M is determined by the function

$$\begin{aligned} \eta_M(f) \colon \mathbb{Z} &\to \widetilde{\Gamma}(M) \\ n &\mapsto [(f \vee n) \wedge (n+1)] - n \end{aligned}$$

Question
Which are the properties of the function $\eta_M(f)$?

Doogood sequences

Definition
A *good pair* in an MMV-algebra A is a pair (x_0, x_1) of elements of A such that $x_0 \oplus x_1 = x_0$ and $x_0 \odot x_1 = x_1$.

A *doogood sequence* in A is a function $\mathrm{x}\colon \mathbb{Z} \to A$ such that

1. definitely for $k \to -\infty$ we have $\mathrm{x}(k) = 1$;
2. definitely for $k \to +\infty$ we have $\mathrm{x}(k) = 0$;
3. for each $k \in \mathbb{Z}$, $(\mathrm{x}(k), \mathrm{x}(k+1))$ is a good pair.

Doogood sequences give quasi-inverse.

Given an MMV-algebra A, we set $\widetilde{\Xi}(A)$ as the set of doogood sequences in A.

$\widetilde{\Xi}(A)$ becomes a unital ℓ-monoid.

$\widetilde{\Xi}$ defines a functor from the category of MMV-algebras to the category of unital ℓ-monoids.

Proposition
$\widetilde{\Gamma}$ and $\widetilde{\Xi}$ are quasi-inverses.

Theorem (Main result)
The category of unital ℓ-monoids is equivalent to the category of MMV-algebras.

Conclusions

Conclusions

Main result
The category of unital ℓ-monoids is equivalent to the category of MMV-algebras.

- No axiom of choice needed to prove the theorem.
- Classical Mundici's equivalence is a consequence (this is shown using the axiom of choice).

Future work

Future work

1. Prove, without the axiom of choice, that the axioms of MMV-algebras hold in any MV-algebra.
   Consequence: proof of classical Mundici's equivalence without axiom of choice.

Future work

2. **Facts**:
   a1. $\langle \mathbb{R}; +, \vee, \wedge, 0 \rangle$ does not generate the variety of ℓ-monoids.
   a2. The variety generated by $\langle \mathbb{R}; +, \vee, \wedge, 0 \rangle$ is not finitely based.
   a3. A countable equational basis for the variety generated by $\langle \mathbb{R}; +, \vee, \wedge, 0 \rangle$ is known.

   To do:
   b1. Prove that $\langle [0, 1]; \oplus, \odot, \vee, \wedge, 0, 1 \rangle$ does not generate the variety of MMV-algebras.
   b2. Prove that the variety generated by $\langle [0, 1]; \oplus, \odot, \vee, \wedge, 0, 1 \rangle$ is not finitely based.
   b3. Provide a countable equational basis for the variety generated by $\langle [0, 1]; \oplus, \odot, \vee, \wedge, 0, 1 \rangle$.

Future work

3. **Fact**:
   The class of $(+, \vee, \wedge, 0)$-subreducts of ℓ-groups is axiomatised by the equations defining ℓ-monoids together with the cancellation law:

$$x + z = y + z \Longrightarrow x = y.$$

   To do:
   Prove that the class of $\{\oplus, \odot, \vee, \wedge, 0, 1\}$-subreducts of MV-algebras is axiomatised by the equations defining MMV-algebras together with the single quasi-equation

$$\text{If } x \oplus z = y \oplus z \text{ and } x \odot z = y \odot z, \text{ then } x = y.$$

Thank you for your attention!