Unital commutative distributive ℓ-monoids and their unit intervals

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DOCToR: Duality, Order, (Co)algebras, Topology, and Related topics

Based on

Chapter 4 of my Ph.D. thesis
"On the axiomatisability of the dual of compact ordered spaces",
University of Milan, 2021,

and

A., Equivalence à la Mundici for commutative lattice-ordered monoids,
Algebra Univers. 82, 45 (2021).

Outline

Unital Abelian ℓ-groups and MV-algebras

Definition

A *lattice-ordered group* (or *ℓ-group*, for short) is an algebra $\langle G; \vee, \wedge, +, -, 0\rangle$ (arities 2, 2, 2, 1, 0) such that

1. $\langle G; +, -, 0\rangle$ is a group;
2. $\langle G; \vee, \wedge\rangle$ is a lattice;
3. $+$ preserves the lattice order (i.e. $x \leq y$ implies $x + z \leq y + z$ and $z + x \leq z + y$).

An ℓ-group is called *Abelian* if $+$ is commutative.

The underlying lattice of an ℓ-group is distributive.

Item 3. can be equivalently replaced by "$+$ distributes over $\vee$", or by "$+$ distributes over $\wedge$".

Unit intervals

Given an Abelian ℓ-group $\mathbf{G}$ and an element $1 \in G$ such that $1 \geq 0$, one equips

$$\Gamma(\mathbf{G}, 1) \coloneqq \{x \in G \mid 0 \leq x \leq 1\}$$

with the operations $\vee$, $\wedge$, 0 and 1 defined by restriction,

$$x \oplus y \coloneqq (x + y) \wedge 1, \qquad x \odot y \coloneqq (x + (-1) + y) \vee 0,$$

and

$$\neg x \coloneqq 1 - x.$$

MV-algebras

Definition [Chang, 1958]

An *MV-algebra* is an algebra $\langle A; \vee, \wedge, \oplus, \odot, 0, 1, \neg \rangle$ (arities 2, 2, 2, 2, 0, 0, 1) such that

1. $\langle A; \oplus, 0 \rangle$ and $\langle A; \odot, 1 \rangle$ are commutative monoids.
2. $\vee$ and $\wedge$ are commutative.
3. $\oplus$ distributes over $\wedge$ and $\odot$ distributes over $\vee$.
4. $x \vee y = (x \odot \neg y) \oplus y$ and $x \wedge y = (x \oplus \neg y) \odot y$.
5. $x \oplus \neg x = 1$ and $x \odot \neg x = 0$.

Given an Abelian ℓ-group $\mathbf{G}$ and an element $1 \in G$ such that $1 \geq 0$, $\Gamma(\mathbf{G}, 1)$ is an MV-algebra.
Every MV-algebra arises in this way, up to iso.

Good $\mathbb{Z}$-sequences

Given an MV-algebra $\mathbf{A}$, we construct the Abelian ℓ-group $\mathbb{G}(\mathbf{A})$ of *good $\mathbb{Z}$-sequences in the MV-algebra $\mathbf{A}$*, i.e., functions $\mathbf{x}\colon \mathbb{Z} \to A$ s.t.

1. $\mathbf{x}(n) = 1$ eventually for $n \to -\infty$,
2. $\mathbf{x}(n) = 0$ eventually for $n \to +\infty$,
3. for all $n \in \mathbb{Z}$,
$$\mathbf{x}(n) \oplus \mathbf{x}(n+1) = \mathbf{x}(n)$$
(or, equivalently, $\mathbf{x}(n) \odot \mathbf{x}(n+1) = \mathbf{x}(n+1)$).

Denote with 1 the good $\mathbb{Z}$-sequence

$$\begin{aligned} \mathbb{Z} &\longrightarrow A \\ n &\longmapsto \begin{cases} 0 & n > 1; \\ 1 & n \leq 1. \end{cases} \end{aligned}$$

We have $\mathbf{A} \cong \Gamma(\mathbb{G}(\mathbf{A}), 1)$.

Mundici's equivalence

A *strong unit* of an Abelian ℓ-group $\mathbf{G}$ is an element $1 \in G$ such that $1 \geq 0$ and such that, for every $x \in G$, there exists $n \in \mathbb{N}_{>0}$ such that

$$n(-1) \leq x \leq n1.$$

Theorem [Mundici, 1986]

The categories

1. of Abelian ℓ-groups with strong unit and unit-preserving homomorphisms, and
2. of MV-algebras and homomorphisms

are equivalent.

Outline

Definition

An *ℓ-monoid* is an algebra $\langle M; \vee, \wedge, +, 0\rangle$ (arities 2, 2, 2, 0) such that

1. $\langle M; +, 0\rangle$ is a monoid;
2. $\langle M; \vee, \wedge\rangle$ is a lattice;
3. $+$ distributes over $\vee$ and $\wedge$.

An ℓ-monoid $\langle M; \vee, \wedge, +, 0\rangle$ is called *distributive* if the lattice $\langle M; \vee, \wedge\rangle$ is distributive, *commutative* if $+$ is commutative.

Distributive ℓ-monoids : ℓ-groups = monoids : groups.

Commutative distributive ℓ-monoids : Abelian ℓ-groups = commutative monoids : Abelian groups.

Given a commutative distributive ℓ-monoid $\mathbf{M}$ and an invertible element $1 \in M$ with $1 \geq 0$, set

$$\Gamma(\mathbf{M}, 1) := \{x \in M \mid 0 \leq x \leq 1\}.$$

We turn $\Gamma(\mathbf{M}, 1)$ into an algebra $\langle \Gamma(\mathbf{M}, 1); \vee, \wedge, \oplus, \odot, 0, 1 \rangle$ (arities 2, 2, 2, 2, 0, 0) by defining $\vee$, $\wedge$, 0 and 1 by restriction, and

$$x \oplus y := (x + y) \wedge 1, \qquad x \odot y := (x + (-1) + y) \vee 0.$$

MV-monoidal algebras

Definition

An *MV-monoidal algebra* is an algebra $\langle A; \vee, \wedge, \oplus, \odot, 0, 1\rangle$ (arities 2, 2, 2, 2, 0, 0) such that

1. $\langle A; \oplus, 0\rangle$ and $\langle A; \odot, 1\rangle$ are commutative monoids.
2. $\langle A; \vee, \wedge\rangle$ is a distributive lattice.
3. Both $\oplus$ and $\odot$ distribute over both $\vee$ and $\wedge$.
4. $(x \oplus y) \odot ((x \odot y) \oplus z) = (x \odot (y \oplus z)) \oplus (y \odot z)$.
5. $(x \odot y) \oplus z = ((x \odot (y \oplus z)) \oplus (y \odot z)) \vee z$, and
   $(x \oplus y) \odot z = ((x \oplus (y \odot z)) \odot (y \oplus z)) \wedge z$.

Unit intervals

Given a commutative distributive ℓ-monoid $\mathbf{M}$ and an invertible element $1 \in M$ such that $1 \geq 0$, the algebra $\Gamma(\mathbf{M}, 1)$ is an MV-monoidal algebra.
Every MV-monoidal algebra arises in this way, up to iso.

Given an MV-monoidal algebra $\mathbf{A}$, one constructs the commutative distributive ℓ-monoid $\mathbb{G}(\mathbf{A})$ of *good $\mathbb{Z}$-sequences*, i.e. functions $\mathbf{x}\colon \mathbb{Z} \to A$ such that

1. $\mathbf{x}(n) = 0$ eventually for $n \to +\infty$,
2. $\mathbf{x}(n) = 1$ eventually for $n \to -\infty$,
3. for all $n \in \mathbb{Z}$, $\mathbf{x}(n) \oplus \mathbf{x}(n+1) = \mathbf{x}(n)$,
4. for all $n \in \mathbb{Z}$, $\mathbf{x}(n) \odot \mathbf{x}(n+1) = \mathbf{x}(n+1)$.

Denote with 1 the good $\mathbb{Z}$-sequence

$$\begin{aligned} \mathbb{Z} &\longrightarrow A \\ n &\longmapsto \begin{cases} 0 & n > 1; \\ 1 & n \leq 1. \end{cases} \end{aligned}$$

We have $\mathbf{A} \cong \Gamma(\mathbb{G}(\mathbf{A}), 1)$.

A *strong unit* of a commutative distributive ℓ-monoid $\mathbf{M}$ is an invertible element $1 \in M$ such that $1 \geq 0$ and such that, for every $x \in M$, there exists $n \in \mathbb{N}_{>0}$ such that

$$n(-1) \leq x \leq n1.$$

Main result

The categories

1. of commutative distributive ℓ-monoids with strong unit and unit-preserving homomorphisms, and
2. of MV-monoidal algebras and homomorphisms

are equivalent.

The proof is choice-free.

Corollary

The categories

1. of Abelian ℓ-groups with strong unit and unit-preserving homomorphisms, and
2. of algebras $\langle A; \vee, \wedge, \oplus, \odot, 0, 1, \neg \rangle$ s.t.
   2.1 $\langle A; \vee, \wedge, \oplus, \odot, 0, 1 \rangle$ is an MV-monoidal algebra,
   2.2 $x \oplus \neg x = 1$ and $x \odot \neg x = 0$

   and homomorphisms

are equivalent.

The proof is choice-free.

Outline

Unital Abelian ℓ-groups, MV-algebras

Unital commutative distributive ℓ-monoids, MV-monoidal algebras

Applications, further work

Compact ordered space

Definition [Nachbin, 1948]

A *compact ordered space* is a compact space X equipped with a partial order $\leq$, closed in $X \times X$.

Given a compact ordered space X,

1. the set of order-preserving continuous functions from X to $\mathbb{R}$ is a commutative distributive ℓ-monoid with strong unit (with pointwise defined operations), and
2. the set of order-preserving continuous functions from X to $[0, 1]$ is an MV-monoidal algebra (with pointwise defined operations).

Theorem [A., 2019, A., Reggio, 2020]

The category of compact ordered spaces and order-preserving continuous maps is dually equivalent to a variety of infinitary algebras.

We provide a finite equational axiomatization.

Axiomatization

Definition

A *2-divisible MV-monoidal algebra* is an algebra $\langle A; \vee, \wedge, \oplus, \odot, 0, 1, \mathrm{h}, \mathrm{j}\rangle$ (arities 2, 2, 2, 2, 0, 1, 1) such that

1. $\langle A; \vee, \wedge, \oplus, \odot, 0, 1\rangle$ is an MV-monoidal algebra;
2. $\mathrm{j}(x) = \mathrm{h}(1) \oplus \mathrm{h}(x)$;
3. $\mathrm{h}(x) = \mathrm{j}(0) \odot \mathrm{j}(x)$;
4. $\mathrm{h}(x) \oplus \mathrm{h}(x) = x$;
5. $\mathrm{j}(x) \odot \mathrm{j}(x) = x$;
6. $\mathrm{h}(\mathrm{h}(x) \oplus \mathrm{h}(y)) = \mathrm{h}(\mathrm{h}(x)) \oplus \mathrm{h}(\mathrm{h}(y))$;
7. $\mathrm{j}(\mathrm{j}(x) \odot \mathrm{j}(y)) = \mathrm{j}(\mathrm{j}(x)) \odot \mathrm{j}(\mathrm{j}(y))$.

Intented interpretation in $[0, 1]$:

$$\mathrm{h}(x) = \frac{x}{2}, \quad \mathrm{j}(x) = \frac{1}{2} + \frac{x}{2}.$$

For $n \in \mathbb{N}$,

$$\tau_n(x, y) \coloneqq \big(x \wedge (y \oplus \mathrm{h}^n(1))\big) \vee \big(y \odot \mathrm{j}^n(0)\big).$$

Inductively on $n \in \mathbb{N}_{>0}$:

$$\mu_1(x_1) \coloneqq x_1;$$
$$\mu_n(x_1, \ldots, x_n) \coloneqq \tau_{n-1}\big(x_n, \mu_{n-1}(x_1, \ldots, x_{n-1})\big).$$

Definition

A *limit 2-divisible MV-monoidal algebra* is an algebra $\langle A; \vee, \wedge, \oplus, \odot, 0, 1, \mathrm{h}, \mathrm{j}, \lambda \rangle$ (arities $2, 2, 2, 2, 0, 0, 1, 1, \omega$) such that

1. $\langle A; \vee, \wedge, \oplus, \odot, 0, 1, \mathrm{h}, \mathrm{j} \rangle$ is a 2-divisible MV-monoidal algebra.
2. $\lambda(x, x, \dots) = x$.
3. $\lambda\big(\tau_0(x, y), \tau_1(x, y), \dots\big) = y$.
4. $\lambda(x_1, x_2, \dots) = \lambda(\mu_1(x_1), \mu_2(x_1, x_2), \dots)$.
5. $\mu_2(x_1, x_2) \odot \mathrm{j}(0) \leq \lambda(x_1, x_2, \dots) \leq \mu_2(x_1, x_2) \oplus \mathrm{h}(1)$.
6. $\lambda(x_1, x_2, \dots) \oplus \lambda(x_1, x_2, \dots) = \lambda\big(\mu_2(x_1, x_2) \oplus \mu_2(x_1, x_2), \mu_3(x_1, x_2, x_3) \oplus \mu_3(x_1, x_2, x_3), \dots\big)$.
7. $\lambda(x_1, x_2, \dots) \odot \lambda(x_1, x_2, \dots) = \lambda\big(\mu_2(x_1, x_2) \odot \mu_2(x_1, x_2), \mu_3(x_1, x_2, x_3) \odot \mu_3(x_1, x_2, x_3), \dots\big)$.

Intended interpretation in $[0, 1]$:

$$\lambda(x_1, x_2, \dots) = \lim_{n \to \infty} \mu_n(x_1, \dots, x_n).$$

Theorem

The category of compact ordered spaces and order-preserving continuous maps is dually equivalent to the (infinitary, finitely axiomatized) variety of limit 2-divisible MV-monoidal algebras.

Positive MV-algebras

From an abstract by Cabrer, Jipsen, Kroupa (SYSMICS 2019):

Definition

A *positive MV-algebra* is an algebra $\langle A; \vee, \wedge, \oplus, \odot, 0, 1\rangle$ that is isomorphic to a subreduct of some MV-algebra.

The class of positive MV-algebras is the quasivariety generated by $\langle [0,1]; \vee, \wedge, \oplus, \odot, 0, 1\rangle$.

Term-functions on $[0,1]$: order-preserving McNaughton functions.

"Positive MV-algs. : MV-algs. = bdd. distr. lattices : Bool. algs."

Question

Does there exist a finite quasi-equational axiomatization for the class of positive MV-algebras?

Proposition [Work in progress with Jipsen, Kroupa, Vannucci]

An algebra $\langle A; \vee, \wedge, \oplus, \odot, 0, 1\rangle$ is a positive MV-algebra iff it is an MV-monoidal algebra satisfying

$$\text{If } x \oplus z = y \oplus z \text{ and } x \odot z = y \odot z, \text{ then } x = y. \tag{$\star$}$$

A commutative distributive ℓ-monoid $\mathbf{M}$ with strong unit 1 is cancellative iff $(\star)$ holds in $\Gamma(\mathbf{M}, 1)$.

Finite non-axiomatizability

An *ℓ-semigroup* is an algebra $\langle M; \vee, \wedge, +\rangle$ such that $\langle M; +\rangle$ is a semigroup, $\langle M; \vee, \wedge\rangle$ is a lattice, and $+$ distributes over $\vee$ and $\wedge$. An ℓ-semigroup $\langle M; \vee, \wedge, +\rangle$ is called *distributive* if the lattice $\langle M; \vee, \wedge\rangle$ is distributive, *commutative* if $+$ is commutative.

Proposition (see [Repnitskii, 1983])

The variety of commutative distributive ℓ-semigroups is not generated by any cancellative commutative distributive ℓ-semigroup.

For example,

$$(x + y) \wedge (z + w) \leq (x + z) \vee (y + w).$$

holds in every cancellative commutative distributive ℓ-semigroup, but fails in some (totally ordered, positive) commutative distributive ℓ-semigroups.

Theorem [Repnitskii, 1983]

The variety generated by any nontrivial cancellative commutative distributive ℓ-semigroup admits no finite equational axiomatization.

Finite non-axiomatizability

Proposition

The variety of MV-monoidal algebras is not generated by $\langle [0,1]; \vee, \wedge, \oplus, \odot, 0, 1\rangle$.

Indeed,

$$(x \oplus y) \wedge (z \oplus w) \leq (x \oplus z) \vee (y \oplus w).$$

holds in $[0,1]$ but fails in some MV-monoidal algebras.

Conjecture

The variety generated by $\langle [0,1]; \vee, \wedge, \oplus, \odot, 0, 1\rangle$ admits no finite equational axiomatization.

Perfect MV-monoidal algebras

We have an equivalence between

1. the category of commutative distributive ℓ-monoids, and
2. a full subcategory of the category of MV-monoidal algebras, whose objects we call *perfect MV-monoidal algebras*.

Thank you for your attention.