Stone-Gelfand duality for groups

Marco Abbadini
Dipartimento di Matematica *Federigo Enriques*
University of Milan, Italy
marco.abbadini@unimi.it

Joint work with Vincenzo Marra and Luca Spada

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Overview

Part I: Classical Stone-Gelfand duality

Part II: Stone-Gelfand duality for groups

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Part I: Classical Stone-Gelfand duality

Part II: Stone-Gelfand duality for groups

Stone duality:

$$\mathsf{Stone}^{\mathrm{op}} \cong \mathsf{Boole}$$

Two ways of reading it.

1. Representation of Boolean algebras (as algebras of sets).
2. Stone is the dual of a variety of finitary algebras.

Variety of algebras: category of τ-algebras (where τ is a set of function symbols) satisfying a certain set of equations.

$$\forall \underline{x} \quad \gamma(\underline{x}) = \eta(\underline{x}).$$

Finitary algebras: every primitive operation has finite arity.

$$\mathsf{CompHaus} \coloneqq \text{category of compact Hausdorff spaces;}$$
$$\text{morphisms: continuous functions.}$$

$$\mathsf{CompHaus}^{\mathrm{op}} \cong \ ?$$

Is $\mathsf{CompHaus}^{\mathrm{op}}$ equivalent to...

- ... a variety of finitary algebras? No
- ... <u>a class of finitary algebras</u>? Yes (Stone-Gelfand duality)
- ... an elementary class? Open question
- ... <u>a variety of (possibly infinitary) algebras</u>? Yes

Given a Stone space S, the Boolean algebra associated to S is

$$\{A \subseteq S \mid A \text{ is clopen}\},$$

or, equivalently (taking charateristic functions),

$$\{f : S \to \{0,1\} \mid f \text{ is continuous}\}.$$

The space

$$\{f : S \to \{0,1\} \mid S \text{ is continuous}\}$$

is closed under every continuous function $\{0,1\}^I \to \{0,1\}$ (I a set). For example

$$\vee\colon \{0,1\}^2 \to \{0,1\};$$

$$\wedge\colon \{0,1\}^2 \to \{0,1\};$$

$$\neg\colon \{0,1\} \to \{0,1\};$$

$$0\colon \{0,1\}^0 \to \{0,1\};$$

$$1\colon \{0,1\}^0 \to \{0,1\}.$$

1. A function $\{0,1\}^I \to \{0,1\}$ is continuous if, and only if, it depends only on finitely many coordinates.
2. Functional completeness: every continuous function $\{0,1\}^I \to \{0,1\}$ is obtained from the projection functions $\pi_i\colon \{0,1\}^I \to \{0,1\}$ by composition with $\vee, \wedge, \neg, 0, 1$.

Given a compact Hausdorff space X, the continuous functions $X \to \{0, 1\}$ "are not enough".

Example

$[0, 1] \to \{0, 1\}$.

They don't separate points, they forget too much about the structure of $[0, 1]$.

$$C(X,[0,1]) \coloneqq \{f\colon X \to [0,1] \mid f \text{ is continuous}\}.$$

Operations: continuous functions $[0,1]^I \to [0,1]$
(depend on at most countably many coordinates).

$\rightsquigarrow \mathsf{CompHaus}^{\mathrm{op}} \cong$ variety of (infinitary) algebras Δ.

$$C(X,\mathbb{R}) \coloneqq \{f\colon X \to \mathbb{R} \mid f \text{ is continuous}\}.$$

$\rightsquigarrow$ Stone-Gelfand duality.

Stone-Gelfand duality

Given X a compact Hausdorff space, which operations can we define on $C(X, \mathbb{R})$?

Here some of them.

1. Pointwise sum $+$.
2. Supremum, or pointwise maximum $\vee$.
3. Infimum, or pointwise minimum $\wedge$.
4. For each $\lambda \in \mathbb{R}$, the scalar multiplication $\lambda \cdot -$.
5. The constant function 0.
6. The constant function 1.

These operations on $C(X, \mathbb{R})$ are "enough" in order to capture the structure of $C(X, \mathbb{R})$ and to recover X from $C(X, \mathbb{R})$.

Which properties are satisfied by $C(X, \mathbb{R})$, endowed with these operations?

Given a set V, endowed with operations $+, \vee, \wedge, \{\lambda \cdot -\}_{\lambda \in \mathbb{R}}, 0, 1$, we say that...

1. ... V is a *vector lattice*, if
   1.1 $\langle V, 0, +, \{\lambda \cdot -\}_{\lambda \in \mathbb{R}} \rangle$ is a vector space;
   1.2 $\langle V, \vee, \wedge \rangle$ is a distributive lattice;
   1.3 the order is *translation invariant*, i.e., for all $f, g, h \in V$,
   $$f \leqslant g \Rightarrow f + h \leqslant g + h;$$
   1.4 the order is *positively homogeneous*, i.e., for all $\lambda \in \mathbb{R}^+$, for all $f, g \in V$,
   $$f \leqslant g \Rightarrow \lambda \cdot f \leqslant \lambda \cdot g.$$

2. ... V is *archimedean*, if

$$\text{for all } f, g \in V \text{ such that } f \geqslant 0 \text{ and } g \geqslant 0, \text{ we have:}$$
$$\text{if, for all } n \in \mathbb{N}, n \cdot f \leqslant g, \text{ then } f = 0.$$

3. ... 1 is a *strong unit*, if

$$\text{for all } f \in V, \text{ there exists } n \in \mathbb{N} \text{ s.t. } -n \cdot 1 \leqslant f \leqslant n \cdot 1,$$

4. … V is *norm-complete* if, defining the "supremum norm" as

$$\|f\| := \inf\{\lambda \in \mathbb{R}^+ \mid -\lambda \cdot 1 \leqslant f \leqslant \lambda \cdot 1\},$$

V is complete in the metric induced by this norm.

We call *norm-complete vector lattice* a set V, endowed with operations $+, \vee, \wedge, \{\lambda \cdot -\}_{\lambda \in \mathbb{R}}, 0, 1$, that satisfies the previous properties, i.e., V is a vector lattice, V is archimedean, 1 is a strong unit, and V is norm-complete.

$C(X, \mathbb{R})$ is a norm-complete vector lattice.

Is it possible to recover the space X from the structure of norm-complete vector lattice of $C(X, \mathbb{R})$?

Yes.
Idea: each element $x \in X$ gives rise to a function

$$\begin{aligned} \mathrm{ev}_x \colon C(X, \mathbb{R}) &\to \mathbb{R} \\ f &\mapsto f(x). \end{aligned}$$

For a norm-complete vector lattice V, we set

$$Max(V) := \{f \colon V \to \mathbb{R} \mid f \text{ respects } +, \vee, \wedge, \{\lambda \cdot -\}_{\lambda \in \mathbb{R}}, 0, 1\}$$

We endow $Max(V)$ with the subspace topology given by the inclusion $Max(V) \subseteq \mathbb{R}^V$.

$Max(V)$ is a compact Hausdorff space.

$\mathsf{CompHaus}$:= category of compact Hausdorff spaces;
morphisms: continuous functions.

$\mathsf{ComplVectLatt}$:= category of norm-complete vector lattices;
morphisms: functions that preserve
$+, \vee, \wedge, \{\lambda \cdot -\}_{\lambda \in \mathbb{R}}, 0, 1.$

Functors:

$$C(-,\mathbb{R})\colon \mathsf{CompHaus}^{\mathrm{op}} \to \mathsf{ComplVectLatt}$$

and

$$\mathit{Max}\colon \mathsf{ComplVectLatt} \to \mathsf{CompHaus}^{\mathrm{op}}.$$

Theorem (Stone-Gelfand duality)

$C(-,\mathbb{R})$ and Max are quasi-inverses.

$$\mathsf{CompHaus}^{\mathrm{op}} \cong \mathsf{ComplVectLatt}.$$

Main contributors: Banaschewski, Gelfand, Kakutani, Neumark, Yosida.

$$\mathsf{CompHaus}^{\mathrm{op}} \cong \mathsf{ComplVectLatt}.$$

Stone-Gelfand duality can be seen in two ways:

1. Representation of norm-complete vector lattices as $C(X, \mathbb{R})$;
2. $\mathsf{CompHaus}$ as the dual of a class of finitary algebras ($\mathsf{ComplVectLatt}$).

   Function symbols: $+, \vee, \wedge, \{\lambda \cdot -\}_{\lambda \in \mathbb{R}}, 0, 1$.

   Not elementary:

   (Unit property)
   For all f, there exists $n \in \mathbb{N}$ s.t. $|f| \leqslant n \cdot 1$.

   (Archimedean property)
   For all f, g such that $f \geqslant 0$ and $g \geqslant 0$, we have:
   if, for all $n \in \mathbb{N}$, $nf \leqslant g$, then $f = 0$.

CompHaus as dual of an infinitary variety

$$C(X,[0,1]) \coloneqq \{f\colon X \to [0,1] \mid f \text{ is continuous}\}.$$

$$\rightsquigarrow \mathsf{CompHaus}^{\mathrm{op}} \cong \text{variety of (infinitary) algebras } \Delta.$$

Operations: continuous functions $[0,1]^I \to [0,1]$
(depend on at most countably many coordinates).

Primitive operations: $\oplus, \neg, 0, \delta$

$$x \oplus y \coloneqq \min\{x+y, 1\}.$$

$$\neg x \coloneqq 1 - x.$$

$$0 \in [0,1].$$

$$\delta(x_1, x_2, \dots) \coloneqq \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

(Duskin, Isbell, Marra, Reggio)

$$\mathsf{CompHaus}^{\mathrm{op}} \cong \mathsf{ComplVectLatt} \cong \Delta$$

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$$\mathsf{CompHaus}^{\mathrm{op}} \cong \mathsf{ComplVectLatt} \cong \Delta,$$

What if we replace the linear structure of norm-complete vector lattices with a weaker one?: structure of abelian group.

$$\text{(Category of spaces)}^{\mathrm{op}} \overset{?}{\cong} \mathsf{Compl}\ell\mathsf{Groups} \overset{?}{\cong} \text{Variety}$$

Given $q \in \mathbb{R}$, the additive subgroup of $\mathbb{R}$ generated by $\{q, 1\}$ is

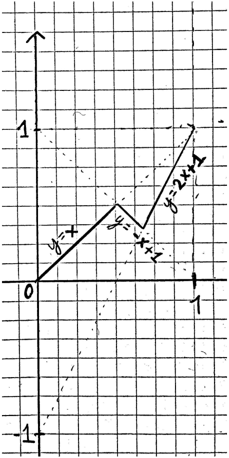
$$\begin{aligned} \text{if } q \in \mathbb{Q},\ \ n = \texttt{den}(q) \rightsquigarrow & \qquad \frac{1}{n}\mathbb{Z}; \\ \text{if } q \in \mathbb{R} \setminus \mathbb{Q} \rightsquigarrow & \qquad \text{dense subset of } \mathbb{R}. \end{aligned}$$

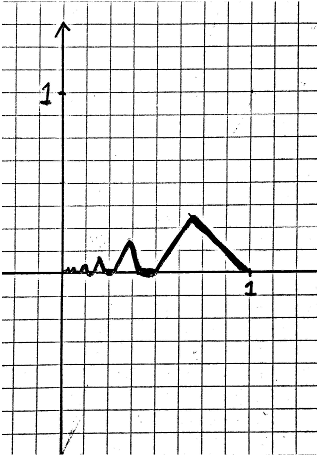
The topological closure $\overline{\langle q, 1\rangle}$ of the additive subgroup

generated by $\{q, 1\}$ is

$$\begin{aligned} \text{if } q \in \mathbb{Q},\ \ n := \texttt{den}(q) \rightsquigarrow & \qquad \frac{1}{n}\mathbb{Z}; \\ \text{if } q \in \mathbb{R} \setminus \mathbb{Q} \rightsquigarrow & \qquad \mathbb{R}. \end{aligned}$$

$$C_{\mathrm{den}}([0,1],\mathbb{R}) :=$$
$$\left\{ f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}, \forall x \in [0,1] \;\; f(x) \in \overline{\langle x, 1 \rangle} \right\}$$





Which operations can we define on $C_{\mathrm{den}}([0,1],\mathbb{R})$?

Here some of them.

1. Pointwise sum $+$.
2. Supremum, or pointwise maximum $\vee$.
3. Infimum, or pointwise minimum $\wedge$.
4. The pointwise opposite $-$.
5. The constant function 0.
6. The constant function 1.

Which properties are satisfied by $C_{\mathrm{den}}([0,1],\mathbb{R})$, endowed with these operations?

Given a set G, endowed with operations $+, \vee, \wedge, -, 0, 1$, we say that...

1. ... G is an *abelian lattice-ordered group*, if
   1.1 $\langle G, 0, +, - \rangle$ is an abelian group;
   1.2 $\langle G, \vee, \wedge \rangle$ is a distributive lattice;
   1.3 the order is translation invariant, i.e., for all $f, g, h \in G$,

$$f \leqslant g \Rightarrow f + h \leqslant g + h.$$

2. ... G is *archimedean*, if

$$\text{for all } f, g \in G \text{ such that } f \geqslant 0 \text{ and } g \geqslant 0, \text{ we have:}$$
$$\text{if, for all } n \in \mathbb{N},\ nf \leqslant g, \text{ then } f = 0.$$

3. ... 1 is a *strong unit*, if

$$\text{for all } f \in G \text{, there exists } n \in \mathbb{N} \text{ s.t. } -n1 \leqslant f \leqslant n1,$$

4. ... G is *norm-complete*, if defining the "supremum norm" as

$$\|f\| := \inf\left\{\frac{p}{q} \in \mathbb{Q}^+ \mid -p1 \leqslant qf \leqslant p1\right\},$$

G is complete in the metric $d(f,g) := \|f - g\|$ induced by this norm.

We call *norm-complete ℓ-group* a set G, endowed with operations $+, \vee, \wedge, -, 0, 1$, that satisfies the previous properties, i.e., G is an abelian lattice-ordered group, G is archimedean, 1 is a strong unit, and G is norm-complete.

$C_{\text{den}}([0,1], \mathbb{R})$ is a norm-complete ℓ-group.

$$\mathsf{Compl}\ell\mathsf{Groups} := \text{category of norm-complete } \ell\text{-groups; morphisms: functions that preserve } +, \vee, \wedge, -, 0, 1.$$

$$(\text{Category of spaces})^{\text{op}} \overset{?}{\cong} \mathsf{Compl}\ell\mathsf{Groups} \overset{?}{\cong} \text{Variety}$$

The category of spaces

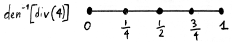
$$\mathtt{den}\colon [0,1] \to \mathbb{N}$$

$$q \mapsto \mathtt{den}(q) := \begin{cases} \text{denominator of } q & \text{if } q \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

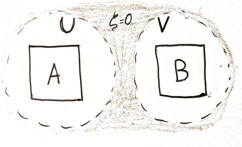
Definition

An *a-normal space* (X, ζ) is a compact Hausdorff space X, endowed with a function $\zeta\colon X \to \mathbb{N}$ such that

1. $\forall n \in \mathbb{N}, \zeta^{-1}[\mathtt{div}(n)]$ is closed;



2. $\forall A, B \subseteq X$ closed and disjoint, $\exists U, V$ open disjoint neighbourhoods of A and B s.t. $\forall x \in X \setminus (U \cup V), \zeta(x) = 0$.



Given (X, ζ) and (X', ζ') a-normal spaces, a *morphism* from (X, ζ) to (X', ζ') is a continuous function $f\colon X \to X'$ that "respects the denominators":

$$\text{for any } x \in X,\ \zeta'(f(x)) \text{ divides } \zeta(x).$$

Example:

$$f\colon ([0,1], \mathtt{den}) \to ([0,1], \mathtt{den})$$
$$f\left(\frac{1}{4}\right) \in \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}.$$

Theorem

The category of norm-complete ℓ-groups is dually equivalent to the category of a-normal spaces.

$$\textsf{a-Normal}^{\mathrm{op}} \cong \textsf{Compl}\ell\textsf{Groups} \overset{?}{\cong} \text{Variety}$$

The variety

Theorem

The category of norm-complete ℓ-groups is equivalent to a variety of (infinitary) algebras.

This variety is denoted with CMV. The objects are "norm-complete MV-algebras".
Operations: continuous functions $[0,1]^I \to [0,1]$ that "respect the denominators".
Primitive operations: $\oplus, \neg, 0, \gamma$.
$\gamma \colon [0,1]^{\mathbb{N}} \to [0,1]$ is a function that calculates the limit of "quickly converging" sequences.

$$\mathsf{a\text{-}Normal}^{\mathrm{op}} \cong \mathsf{Compl}\ell\mathsf{Groups} \cong \mathsf{CMV}$$

Conclusions

$$
\begin{array}{ccccc}
\mathsf{CompHaus}^{\mathrm{op}} & \cong & \mathsf{ComplVectLatt} & \cong & \Delta \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathsf{a\text{-}Normal}^{\mathrm{op}} & \cong & \mathsf{Compl}\ell\mathsf{Groups} & \cong & \mathsf{CMV}
\end{array}
$$

Thank you for your attention.