TUTORATO LOGICA MATEMATICA
A.A. 2022/2023

ESERCIZI 2022.11.17

Esercizio 1. Formalizzare al prim'ordine la classe dei gruppi.

Esercizio 2. Fornire un'assiomatizzazione al prim'ordine dei gruppi privi di torsione.

Soluzione. Per ogni $n \in \mathbb{N}$, prendiamo il seguente assioma.

$$\forall x(\neg(x = 1) \to \neg(\underbrace{x \cdot \dots \cdot x}_{n \text{ volte}} = 1))$$

□

Esercizio 3. Utilizzando le regole della deduzione naturale, produrre derivazioni per i seguenti fatti (le lettere x, y, z sono variabili, le lettere a, b, c sono costanti):

(1) $\vdash \forall x \neg(F(x) \wedge \neg F(x))$.
(2) $R(a), \forall x(R(x) \to S(x)) \vdash \exists x S(x)$.
(3) $\exists x R(x), \forall x(R(x) \to S(x)) \vdash \exists x S(x)$.
(4) $\forall x R(x) \vdash \forall y R(y)$.
(5) $\exists x R(x) \vdash \exists y R(y)$.
(6) $\neg \exists x \neg R(x) \vdash \forall x R(x)$.
(7) $\neg \forall x R(x) \vdash \exists x \neg R(x)$.
(8) $\exists x \neg R(x) \vdash \neg \forall x R(x)$.
(9) $\exists x \exists y R(x, y) \vdash \exists y \exists x R(x, y)$.
(10) $\forall x(F(x) \to G(a)) \vdash (\exists x F(x)) \to G(a)$.
(11) $\vdash \exists x(R(x) \to \forall y R(y))$ (in ["Logic and Structure", van Dalen], è scritto che è istruttivo pensare a $R(x)$ come "x beve").
(12) $(\exists x F(x)) \to G(a) \vdash \forall x(F(x) \to G(a))$.
(13) $\exists x(P \to R(x)) \vdash P \to \exists x R(x)$.
(14) $\exists x \forall y A(x, y) \vdash \forall y \exists x A(x, y)$.
(15) $\vdash \exists x \exists y(R(x, y) \to R(y, x))$.
(16) $\forall x(F(x) \vee \neg F(x))$.
(17) $\forall x F(x) \wedge \forall x G(x) \vdash \forall x(F(x) \wedge G(x))$.
(18) $\forall x \exists y \forall z R(x, y, z) \vdash \forall x \forall z \exists y R(x, y, z)$.
(19) $\forall x \forall y R(x, y) \vdash \forall x(R(x, x) \wedge \forall y R(y, x))$.
(20) $\forall x \forall y R(x, y) \vdash \forall x \forall y(R(x, y) \wedge R(y, x))$.
(21) $\exists x P(x) \vee \exists y Q(y) \vdash \exists z(P(z) \vee Q(z))$.
(22) $\forall x(\exists y P(y) \to Q(x)) \vdash \forall x \exists y(P(y) \to Q(x))$.
(23) $\forall x \neg \forall y(P(x, y) \to Q(x, y)) \vdash \forall x \exists y P(x, y)$.
(24) $\neg \forall x \neg \forall y R(y, x) \vdash \forall x \neg \forall y \neg R(x, y)$.

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(25) $\forall x(F(x) \to G(x)), \forall x F(x) \vdash \exists x G(x)$.
(26) $\forall x(F(x) \to \neg G(x)), \exists x G(x) \vdash \exists x \neg F(x)$.

Soluzione. (1).

$$
\begin{prooftree}
\AxiomC{$[F(x) \land \neg F(x)]^1$}
\LeftLabel{E$\land$}
\UnaryInfC{$F(x)$}
\AxiomC{$[F(x) \land \neg F(x)]^1$}
\LeftLabel{E$\land$}
\UnaryInfC{$\neg F(x)$}
\LeftLabel{E$\neg$}
\BinaryInfC{$\bot$}
\LeftLabel{I$\neg_1$}
\UnaryInfC{$\neg(F(x) \land \neg F(x))$}
\LeftLabel{I$\forall$}
\UnaryInfC{$\forall x \neg(F(x) \land \neg F(x))$}
\end{prooftree}
$$

(2)

$$
\begin{prooftree}
\AxiomC{$\forall x(R(x) \to S(x))$}
\LeftLabel{E$\forall$}
\UnaryInfC{$R(a) \to S(a)$}
\AxiomC{$R(a)$}
\LeftLabel{E$\to$}
\BinaryInfC{$S(a)$}
\LeftLabel{I$\exists$}
\UnaryInfC{$\exists x S(x)$}
\end{prooftree}
$$

(3)

$$
\begin{prooftree}
\AxiomC{$\exists x R(x)$}
\AxiomC{$[R(x)]^1$}
\AxiomC{$\forall x(R(x) \to S(x))$}
\LeftLabel{E$\forall$}
\UnaryInfC{$R(x) \to S(x)$}
\LeftLabel{E$\to$}
\BinaryInfC{$S(x)$}
\LeftLabel{E$\exists_1$}
\BinaryInfC{$\exists x S(x)$}
\end{prooftree}
$$

(4)

$$
\begin{prooftree}
\AxiomC{$\forall x R(x)$}
\LeftLabel{E$\forall$}
\UnaryInfC{$R(y)$}
\LeftLabel{I$\forall$}
\UnaryInfC{$\forall y R(y)$}
\end{prooftree}
$$

(5)

$$
\begin{prooftree}
\AxiomC{$\exists x R(x)$}
\AxiomC{$[R(x)]^1$}
\LeftLabel{I$\exists$}
\UnaryInfC{$\exists y(R(y))$}
\LeftLabel{E$\exists_1$}
\BinaryInfC{$\exists y R(y)$}
\end{prooftree}
$$

(6)

$$
\begin{prooftree}
\AxiomC{$[\neg R(x)]^1$}
\LeftLabel{I$\exists$}
\UnaryInfC{$\exists x \neg R(x)$}
\AxiomC{$\neg \exists x \neg R(x)$}
\LeftLabel{E$\neg$}
\BinaryInfC{$\bot$}
\UnaryInfC{$R(x)$}
\LeftLabel{I$\forall_1$}
\UnaryInfC{$\forall x R(x)$}
\end{prooftree}
$$

(7)

$$
\begin{prooftree}
\AxiomC{$[\neg\exists\neg R(x)]^1$}
\AxiomC{$[\neg R(x)]^2$}
\LeftLabel{E$\exists$}
\UnaryInfC{$\exists x\neg R(x)$}
\LeftLabel{E$\neg$}
\BinaryInfC{$\perp$}
\LeftLabel{RA$_2$}
\UnaryInfC{$R(x)$}
\LeftLabel{I$\forall$}
\UnaryInfC{$\forall x R(x)$}
\AxiomC{$\neg\forall x R(x)$}
\LeftLabel{E$\neg$}
\BinaryInfC{$\perp$}
\LeftLabel{RA$_1$}
\UnaryInfC{$\exists x\neg R(x)$}
\end{prooftree}
$$

(8)

$$
\begin{prooftree}
\AxiomC{$\exists x\neg R(x)$}
\AxiomC{$\neg R(x)$}
\AxiomC{$[\forall x R(x)]^1$}
\LeftLabel{E$\forall$}
\UnaryInfC{$R(x)$}
\LeftLabel{E$\neg$}
\BinaryInfC{$\perp$}
\LeftLabel{E$\exists$}
\BinaryInfC{$\perp$}
\LeftLabel{I$\neg_1$}
\UnaryInfC{$\neg\forall x R(x)$}
\end{prooftree}
$$

(9)

$$
\begin{prooftree}
\AxiomC{$\exists x\exists y R(x,y)$}
\AxiomC{$[\exists y R(x,y)]_1$}
\AxiomC{$[R(x,y)]^2$}
\LeftLabel{I$\exists$}
\UnaryInfC{$\exists x R(x,y)$}
\LeftLabel{I$\exists$}
\UnaryInfC{$\exists y\exists x R(x,y)$}
\LeftLabel{E$\exists$}
\BinaryInfC{$\exists y\exists x R(x,y)$}
\LeftLabel{E$\exists_2 1$}
\BinaryInfC{$\exists y\exists x R(x,y)$}
\end{prooftree}
$$

(10)

$$
\begin{prooftree}
\AxiomC{$[\exists x F(x)]^1$}
\AxiomC{$[F(x)]^2$}
\AxiomC{$\forall x(F(x)\to G(a))$}
\UnaryInfC{$F(x)\to G(a)$}
\BinaryInfC{$G(a)$}
\BinaryInfC{$G(a)$}
\LeftLabel{I$\to$}
\UnaryInfC{$(\exists x F(x))\to G(a)$}
\end{prooftree}
$$

□