CATEGORICAL DUALITIES IN LOGIC

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Every Wed, in this room, until 8 April

Exam: 19 June (Oral)

https://marcoabbadini-uni.github.com → Teaching → Categorical Dualities in Log.

$$2 = 2 \qquad\qquad \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$$

EASIER

CLASSICAL PROP. LOGIC $\quad \underbrace{\text{Boolean algebras}}_{\text{syntax}} \simeq^{op} \boxed{\text{Stone Spaces}} \simeq$ spaces of models

categorical duality

semantics

$\simeq$ algebras of formulas

→ INTUITIONISTIC

→ FIRST-ORDER

CLASSICAL PROPOSITIONAL LOGIC

- propositional language $\mathcal{L}$, i.e. a set (of propositional symbols / variables): p, q, r

- connectives: $\vee, \wedge, \neg, 0, 1$

$\vee$: or, join; $\wedge$: and, meet; $\neg$: not; 0: false, bottom ($\bot$); 1: True, top ($\top$)

$$x \Rightarrow y := \neg x \vee y$$

$$x \Leftrightarrow y := (x \rightarrow y) \wedge (y \rightarrow x)$$

- The set of formulas Form($\mathcal{L}$) in the language $\mathcal{L}$ is defined by induction
  - all prop. symb. (i.e., elements of $\mathcal{L}$) are formulas
  - if $\varphi, \psi \in \text{Form}(\mathcal{L})$ then $(\varphi \vee \psi) \in \text{Form}(\mathcal{L})$ and $(\varphi \wedge \psi) \in \text{Form}(\mathcal{L})$
  - if $\varphi \in \text{Form}(\mathcal{L})$, then $\neg\varphi \in \text{Form}(\mathcal{L})$
  - $0, 1 \in \text{Form}(\mathcal{L})$

E.g: p, q, r

$p \wedge q$

$(p \vee q) \wedge \neg q$

$\vdots$

Ex: Consider the language $\mathcal{L} = \{p, q\}$.

The formulas $p \vee q$, $q \vee p$ are equivalent, because they have the same Truth Tables

p	0	1	0	1
q	0	0	1	1
$p \vee q$	0	1	1	1
$q \vee p$	0	1	1	1

Def (Semantic equivalence)

Given a propositional language $\mathcal{L}$, $\varphi, \psi \in \mathrm{Form}(\mathcal{L})$

$$\varphi \equiv \psi \iff \forall v : \mathcal{L} \to \{0,1\} \quad \bar{v}(\varphi) = \bar{v}(\psi)$$

$\equiv$: semantically equivalent

see def. below

Given $v : \mathcal{L} \to \{0,1\}$, we define

$\bar{v} : \mathrm{Form}(\mathcal{L}) \longrightarrow \{0,1\}$ by induction on the complexity of formulas

$$\mathcal{L} \ni p \longmapsto p$$

$$\neg\varphi \longmapsto \neg\,\bar{v}(\varphi) \in \{0,1\}$$

$$\varphi \vee \psi \longmapsto \bar{v}(\varphi) \vee \bar{v}(\psi) \in \{0,1\}$$

$$0 \longmapsto 0$$

$$1 \longmapsto 1$$

$$\varphi \wedge \psi \longmapsto \bar{v}(\varphi) \wedge \bar{v}(\psi)$$

In $2 := \{0,1\}$, by convention,

$0 \wedge 0 = 0$ $\quad 0 \vee 0 = 0$ $\quad \neg 0 = 1$

$0 \wedge 1 =$ $\quad 0 \vee 1 = 1$ $\quad \neg 1 = 0$

$1 \wedge 0 = 0$ $\quad 1 \vee 0 = 1$

$1 \wedge 1 = 1$ $\quad 1 \vee 1 = 1$

For example,

given $v : \begin{array}{l} p \mapsto 1 \\ q \mapsto 0 \end{array}$, we have

$$\bar{v}(p \vee q) = \bar{v}(p) \vee \bar{v}(q) = v(p) \vee v(q) = 1 \vee 0 = 1$$

Example: if $\mathcal{L} = \{p\}$, then $\mathrm{Form}(\mathcal{L})/_{\equiv} = \{[0], [p], [\neg p], [1]\}$

$[1] = [p \vee \neg p]$

For example, if $\mathcal{L} = \{p, q\}$, then one can show that $\left|\mathrm{Form}(\mathcal{L})/_{\equiv}\right| = 2^{2^2} = 16$

Def A **(propositional) theory** T in a (propositional) language $\mathcal{L}$ is a subset of $\mathrm{Form}(\mathcal{L})$.

E.g.: $\mathcal{L} = \{p, q\}$ $\quad T = \{p \vee q\}$

Then, new formulas become equivalent "$p \vee q \equiv_T 1$" "$p \vee \neg q \equiv_T p$"

Def A **model** of a prop. theory T in a lang. $\mathcal{L}$ is a function $v : \mathcal{L} \to \{0, 1\}$ s.t. for all $\sigma \in T$, $\bar{v}(\sigma) = 1$

Def (Semantic equiv. modulo a theory)

Given a theory T in a lang. $\mathcal{L}$, $\varphi, \psi \in Form(\mathcal{L})$

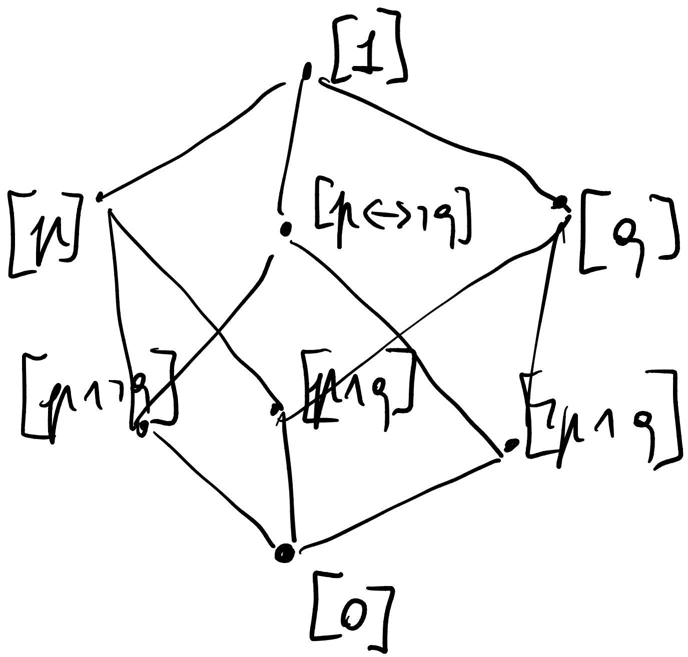
We write

$$\varphi \equiv_T \psi \qquad \text{(semant. equiv. modulo } T\text{)}$$

if for every model $v : \mathcal{L} \to \{0,1\}$ of T, $\bar{v}(\varphi) = \bar{v}(\psi)$.

Example: $\mathcal{L} = \{p, q\}$ $\quad T = \{p \vee q\}$

$Form(\mathcal{L})/_{\equiv_T}$



$\varphi \leq \psi$ means φ implies ψ

Boolean algebras $\to (B; \vee, \wedge, \neg, 0, 1)$ satisf. certain axioms: $a \wedge \neg a = 0$

$a \vee \neg a = 1$

set

$\hookrightarrow$ can be seen as certain posets (partially ordered sets)

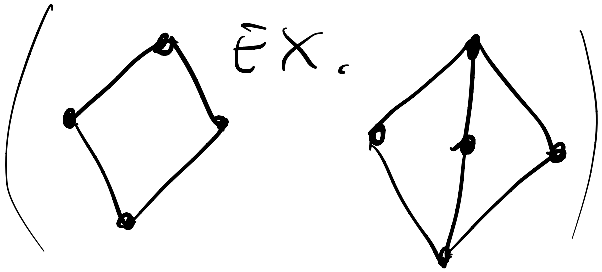
$$p \wedge q \le p$$
$$\le q$$

"$\wedge$" can be interpreted as an infimum

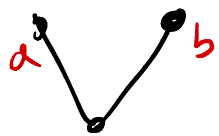
$$r \le p, \ r \le q \Rightarrow r \le p \wedge q$$

Def A lattice (Fr: Treillis) is a poset in which any two elements have a sup and an inf

EX.



NON-EX



a, b do not have a supremum

EX: $(\mathcal{P}(X), \subseteq)$ is a lattice

infimum $A \wedge B := A \cap B$

suprem $A \vee B := A \cup B$

Equiv. def

A lattice is a set L with two binary operations $\vee, \wedge$ satisfying

- $\vee$ and $\wedge$ are comm. and associative
- $\forall a, b \quad a \vee (a \wedge b) = a$
- $\forall a, b \quad a \wedge (a \vee b) = a$.

$\forall x_1 \ldots \forall x_n \; t_1(\ldots) = t_2(\ldots)$

$a \leq b \iff a \wedge b = a$ (equiv. $a \vee b = b$)

Def A bounded lattice (Fr: Treillis borné) is a lattice with a smallest elem. $(0, \text{or } \perp)$ and a greatest elem. $(1, \text{or } \top)$

i.e.

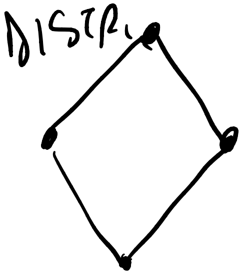
$\forall a \quad a \wedge 0 = 0 \qquad (0 \leq a)$

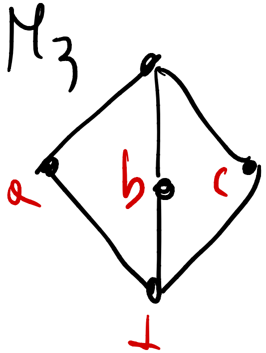
$\forall a \quad a \vee 1 = 1 \qquad (a \leq 1)$

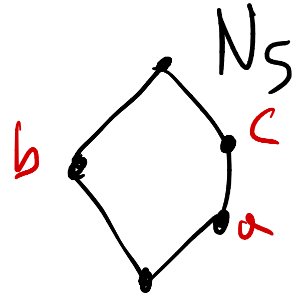
Def A lattice is **distributive** if any of the following equiv. cond. hold:

- $\forall a,b,c \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- $\forall a,b,c \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Ex: $\mathcal{P}(X)$.







$$\underbrace{a \vee \overset{\perp}{(b \wedge c)}}_{a} \neq \underbrace{(a \vee b) \wedge (a \vee c)}_{\top}$$

Thm A lattice is distr. iff it does not have M_3 or N_5 as a sublattice

$$M_3 \overset{\vee,\wedge}{\not\hookrightarrow} L \qquad N_5 \overset{\vee,\wedge}{\not\hookrightarrow} L$$

i.e. there is no injective function from M_3 or N_5 that preserves $\vee, \wedge$

Def A **Bool. alg.** is an alg. structure $\langle B; \vee, \wedge, \neg, 0, 1\rangle$ s.t.

set; bin. op.; neg.; elements of B

- $\langle B; \vee, \wedge, 0, 1\rangle$ is a bounded distr. lattice
- $\forall a \quad a \wedge \neg a = 0$
- $\forall a \quad a \vee \neg a = 1$

"$\neg a$ is a **complement** of a"

they are all equational axioms "$\forall x_1, \dots, \forall x_n$ ---- = ----"

One can show that, for any theory T in a propositional language $\mathcal{L}$

$$\left(\mathrm{Form}(\mathcal{L}) / \equiv_T \; ; \; \underbrace{\vee, \wedge, \neg, 0, 1}\right)$$ is a Bool. alg.

$\equiv_T$: semantic equiv. mod T

these are defined by (they are well-defined)

$$\begin{cases} [\varphi] \vee [\psi] = [\varphi \vee \psi] \\ [\varphi] \wedge [\psi] = [\varphi \wedge \psi] \\ \neg[\varphi] = [\neg\varphi] \\ 1 = [1] \\ 0 = [0] \end{cases}$$

E.g.: $[\varphi] \wedge \neg[\varphi] \overset{?}{=} [0]$ $\qquad \varphi \in \mathrm{Form}(\mathcal{L}_0)$

$$\begin{array}{c} [\varphi] \wedge \neg[\varphi] \\ \| \\ [\varphi \wedge \neg\varphi] \end{array}$$

$$\varphi \wedge \neg\varphi \overset{?}{\equiv}_T 0$$

For every model $v : \mathcal{L}_0 \to \{0,1\}$ of T

$$\bar{v}(\varphi \wedge \neg\varphi) \overset{?}{=} \bar{v}(0)$$

$$\begin{array}{c} \| \\ \bar{v}(\varphi) \wedge \neg\bar{v}(\varphi) \\ \in \{0,1\} \end{array}$$

in $\{0,1\}$ for any $a \in \{0,1\}$ $a \wedge \neg a = 0$

is True because $a \wedge \neg a = 0$ holds in $\{0,1\}$.

All axioms of Bool. alg. hold in $\mathrm{Form}(\mathcal{L}_0)/_{\equiv_T}$ because they are equations holding in $\{0,1\}$

Every equat. that holds in $\{0,1\}$ holds in $\mathrm{Form}(\mathcal{L}_0)/_{\equiv_T}$

(Equation: $\forall x_1, \dots x_n \quad t_1(\dots) = t_2(\dots)$)

- Do Bool. alg. really axiomatize the algebraic structures of the form

$$\mathrm{Form}(\mathcal{L}_0)/_{\equiv_T} \; ?$$

(Or are we missing some axioms?)

Is every Bool. alg. isomorphic to $\mathrm{Form}(\mathcal{L}_0)/_{\equiv_T}$ for some T and $\mathcal{L}_0$?

- Does a Bool. alg. satisfy all equations satisfied by $\{0,1\}$?

YES, (these are consequences of Stone's Repr. Theorem).

Given a set X, the power set $\langle \mathcal{P}(X), \cup, \cap, {}^c, \emptyset, X \rangle$ is a Bool. alg.

$$\subseteq$$

Let's verify that some axioms hold:

$$A \cap A^c = \emptyset$$

$$A \cup A^c = X$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$\supseteq$ easy (it holds in any lattice)

$\subseteq$ let $x \in A \cap (B \cup C)$

$\rightsquigarrow x \in A$

$\rightsquigarrow (x \in B)$ or $(x \in C)$

$\rightsquigarrow (x \in A \cap B)$ or $(x \in A \cap C)$

$\longrightarrow x \in (A \cap B) \cup (A \cap C)$

Also any $\underbrace{\text{subset of } \mathcal{P}(X) \text{ that is closed under } \cup, \cap, {}^c, \emptyset, X}$ is a Bool. alg.

"(Boolean) subalgebra of $\mathcal{P}(X)$"

$\boxed{\text{Stone's Repr. Theorem for Bool. algebras (1936)}}$

Any Boolean algebra is isomorphic to a subalgebra of the power set of some set X.

$\wedge = \cap$

$\vee = \cup$

$\neg = {}^c$

$0 = \emptyset$

$1 = X$

i.e., there is a set X and $B \hookrightarrow \mathcal{P}(X)$.

inj. map

preserves $\vee, \wedge, \neg, 0, 1$

CONSEQ. • It gives a repr. of Bool. alg.: intuition

$$B \hookrightarrow \mathcal{P}(X) = \underbrace{2 \times 2 \times 2 \times \dots}_{|X| \text{ Times}} = 2^X = \prod_{x \in X} 2$$

Equations are preserved by prod. and subalg.

• $\{\text{Boolean algebras}\} = \mathbb{I}\, \mathbb{S}\, \mathbb{P}(2) = \mathbb{H}\mathbb{I}\, \mathbb{S}\, \mathbb{P}(2)$

$\Downarrow$

all equation holding in 2 hold

We give a proof of Stone repr. Theorem.

Stone's Repr. Theorem for Bool. algebras (1936)

Any Boolean algebra is isomorphic to a subalgebra of the power set of some set X.

- Idea

$$B \simeq \mathrm{Form}(\mathcal{L})/\equiv_T \qquad\qquad B \xrightarrow{?} \mathcal{P}(X)$$

$$b \rightsquigarrow [\varphi] \qquad\qquad b \longmapsto Y \subseteq X$$

$$Y = \text{Models of } T$$

we identify a formula with the set of models in which it holds

A model of a prop. theory T in a lang. $\mathcal{L}$ is a valuation $v : \mathcal{L} \to \{0,1\}$ s.t. for all $\sigma \in T$ $\bar{v}(\sigma) = 1$

$$\mathcal{L} \subseteq \mathrm{Form}(\mathcal{L}) \xrightarrow{\bar{v}} \{0,1\}$$

$$\bar{v}(\varphi \wedge \psi) = \bar{v}(\varphi) \wedge \bar{v}(\psi)$$

this corresponds to Bool. hom. $B \longrightarrow 2$ "$\{0,1\}$"

Idea for Stone's repr. Theorem. :

$$X = \hom(B, 2) \quad \text{set of hom. from } B \text{ to } 2$$

Def A **Bool. hom.** $f: A \to B$ between Bool. alg. is a function s.t.

$$f(1) = 1$$
$$f(a \wedge b) = f(a) \wedge f(b)$$
$$f(\neg a) = \neg f(a)$$

(as a conseq. $f(0) = 0$; $f(a \vee b) = f(a) \vee f(b)$)

A homomorph. from B to 2 consists of assigning to each element of B a truth value (0 or 1) in a consistent way, i.e., for example

if $b \mapsto 0$

then $\neg b \mapsto 1$

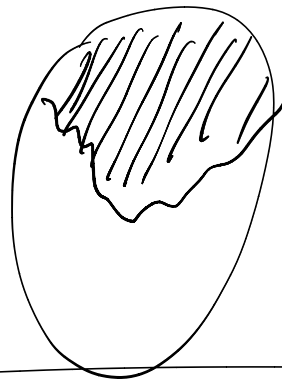
We can identify a Bool. hom. $f : B \to \{0,1\}$ with $f^{-1}[\{1\}]$

The subsets of B of the form $f^{-1}[\{1\}]$ for some hom. $f : B \to 2$ are the "ultrafilters".

Def A ==filter== of a Bool. alg. B is a subset $F \subseteq B$ s.t.

- F is upwards closed ($\overset{x \leq y}{x \in F} \Rightarrow y \in F$)
- $1 \in F$
- $x, y \in F \Rightarrow x \wedge y \in F$

$\left.\begin{matrix} 1 \in F \\ x, y \in F \Rightarrow x \wedge y \in F \end{matrix}\right\}$ F is closed under finite meets



Def An ==ultrafilter== of a Bool. alg. B is a filter F s.t. any of the following equivalent conditions hold:

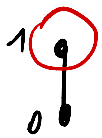
- $\forall x \in B$ exactly one between x and $\neg x$ belongs to F.
- a maximal element in the poset of $\boxed{\text{proper}}$ filters (ordered by inclusion)
  $\hookrightarrow 0 \notin F$
- $B \setminus F$ is an $\boxed{\text{ideal}}$ $\rightsquigarrow$ a downwards closed subset that is closed under finite joins ($x, y \in I \Rightarrow x \vee y \in I$; $0 \in I$

i.e. $x \vee y \in F \Rightarrow x \in F$ or $y \in F$

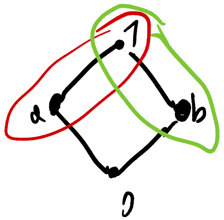
$0 \notin F$

Examples of ultrafilters.

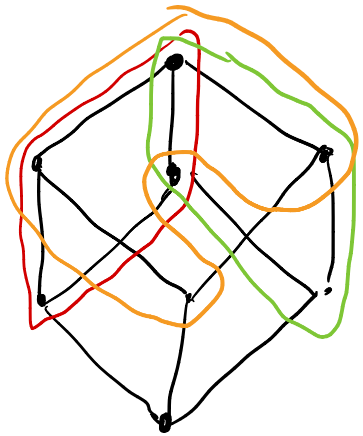
has no ultrafilter



has exactly one ultrafilter: $\{1\}$



has exactly two ultrafilters: $\{a, 1\}$, $\{b, 1\}$



has exactly three ultrafilters.

In a finite Bool. algebra B, every ultrafilter is principal, i.e. it is of the form

$$\uparrow a := \{x \in B \mid a \leq x\},$$

for some $a \in B$.