RECAP

Algebras of formulas $\longleftrightarrow$ Spaces of models

Class. prop. log. Bool. algs $\underset{\text{cat. dual.}}{\longleftrightarrow}$ Stone spaces

Baby version of a cat. duality: Repr. Theorem

Bool. alg.: $\langle B; \vee, \wedge, \neg, 0, 1 \rangle$ $\to$ bounded distr. lattice
arities: 2 2 1 0 0

lattice $\to$ poset $\to$ has finite suprema: $\vee$, 0
$\to$ " infima: $\wedge$, 1

distr.: $\forall a,b,c$ $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

" $\vee$ $\wedge$ $\vee$ $\wedge$ $\vee$

$\to$ for all $a \in B$, $\neg a$ is a complement of a

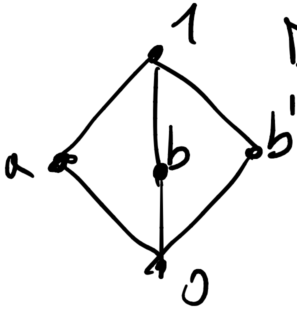
"b is a complement of a iff $a \vee b = 1$, $a \wedge b = 0$"

A Bool. alg. is completely determined by its partial order

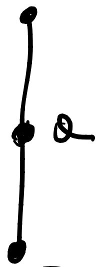
EX In a bounded distr. lattice L, each elem. has at most one complement:

i.e.: for all $a, b, b' \in L$,

$$\left.\begin{array}{l} a \vee b = 1 \\ a \wedge b = 0 \\ a \vee b' = 1 \\ a \wedge b' = 0 \end{array}\right\} \Rightarrow b = b'$$



DISTR. IS NEEDED



(THERE MIGHT NOT EXIST A COMPL. IN A BOUNDED DISTRIBUTIVE LATTICE)

this is a bounded distrib. lattice (any chain is a distr. lattice)

EXAMPLE of Bool. alg.

$\mathcal{P}(X)$, for every set X.

To check that it is a Bool. alg., it suffices to notice that $\mathcal{P}(X) \cong \prod_{x \in X} \mathbf{2}$, that 2 is a Bool. alg., and that Bool. algs are closed under products (and isomorphisms), by the easy direction of Birkhoff's Theorem.

EXAMPLE OF BOOL. ALG. THAT IS NOT A POWERSET

Take X any infinite set, and take

$$FC(X) := \{Y \subseteq X \mid Y \text{ is finite or } \overbrace{\text{cofinite}}^{X \setminus Y \text{ is finite}}\}.$$

This is a Boolean alg. (with $\subseteq$ as partial order), because it is closed under $\cup, \cap, \neg, \emptyset, X$, i.e. because

$FC(X)$ is a subalgebra of $\mathcal{P}(X)$.

Def A (Boolean) <u>subalgebra</u> of a Bool. alg. B is a subset $A \subseteq B$ s.t.

- $0, 1 \in A$
- $x, y \in A \Rightarrow x \vee y \in A$
- $x \in A \Rightarrow \neg x \in A$

REM A subalg. of a Bool. alg. is a Bool. alg. (By the easy direction of Birkhoff's theorem)

To observe that $FC(X)$ is not isomorphic to any powerset, we may observe that $FC(X)$ is not complete: indeed we can write

$$X = \underbrace{Y}_{\text{infinite}} \sqcup \underbrace{Z}_{\text{infinite}},$$ and then

$\{\{y\} \mid y \in Y\}$ has no suprema in $FC(X)$.

Instead, $\mathcal{P}(Y)$ is complete for any set Y.

If X is countable (meaning of the cardinality of $\mathbb{N}$), another way to argue that $FC(X)$ is not isomorphic to any powerset is by cardinality:

- $FC(X)$ is countable
- No power set is countable.

We have seen that every subalgebra of a power set is a Bool. alg.
Stone's Repr. thm says that all Bool. algebras are of this form.

Stone Repr. Thm (1936)

$\boxed{\text{Every Bool. alg. } B \text{ is isom. to a subalg. of } \mathcal{P}(X) \text{ for some set } X.}$

(We will soon see the proof)

Idea: think of el. of B as formulas
think of X = set of models.

↳ Take $X = \mathrm{hom}(B, 2)$

Equivalently, we can take the set of ultrafilters of B.

Def A Bool. hom. $A_1 \to A_2$ is a function s.t.

$f(1) = 1$
$f(x \wedge y) = f(x) \wedge f(y)$
$f(\neg x) = \neg f(x)$

$\Big(\Rightarrow \quad f(0) = 0$
$f(x \vee y) = f(x) \vee f(y) \Big)$

ULTRAFILTER = $\underbrace{\text{FILTER}}$ WHOSE COMPLEMENT $\underbrace{\text{IDEAL}}$

- ↑-closed
- closed under $\wedge, 1$

↓-closed
closed under $\vee, 0$

EXERC. Given a Bool. alg. B, there is a bijection

$$\hom(B,2) \longleftrightarrow \mathrm{Ult}(B)$$

$$f: B \to 2 \longmapsto f^{-1}[\{1\}]$$

$$\left(\chi_U : B \to 2,\quad a \mapsto \begin{cases} 1 & \text{if } a \in U \\ 0 & \text{if } a \notin U \end{cases}\right) \longleftarrow U$$

EX. If B is a $\boxed{\text{finite}}$ Bool. alg., there is a bijection

$$\mathrm{Ult}(B) \longleftrightarrow \{\text{atoms of } B\}$$

$$\uparrow a \longleftarrow a$$

$$U \longmapsto \min U$$

it exists because U is finite (and closed under finite meets)

NO HISTORICAL ACCURACY CLAIMED

Def (Leucippus, Democritus, 5^{th} cent. BC)
An **atom** is a MINIMAL NON-NULL thing.

Def An **atom** of a Bool. Alg. is an element $a \in B$ s.t.

- (NON-NULL) $a > 0$ $\left(\begin{matrix} a \geq 0 \\ a \neq 0 \end{matrix}\right)$
- (MINIMAL) There is no $b \in B$ s.t. $0 < a < b$.

In general, for a (possibly infinite) Bool. alg., we have an injection

$$\mathrm{Ult}(B) \longleftarrow \{\text{atoms of } B\}$$

$$\uparrow a \longleftarrow a$$

which may fail to be surjective, i.e. there are ultrafilters that without a minimum.

(One can show that if an ultrafilter of a Bool. alg. B is of the form $\uparrow a$, then a is an atom)
the ultrafilters of the form $\uparrow a$ for some element $a \in B$ (which must necessarily be an atom)
are called PRINCIPAL ULTRAFILTERS

In $\mathcal{P}(X)$, the atoms are the singletons $\Rightarrow$ for any $x \in X$, $\uparrow\{x\}$ is an ultrafilter,

$$\uparrow\{x\} = \{Y \subseteq X \mid x \in Y\}$$

(As a consequence of the "Boolean Prime Ideal theorem", that we will prove later on in this lecture, whenever X is infinite, $\mathcal{P}(X)$ has some non-principal ultrafilters.
However, this is a nontrivial fact, because the Boolean Prime Ideal Theorem is nontrivial)

To see another example of a nonprincipal

ultrafilter of $FC(\mathbb{N})$

- For every $n \in \mathbb{N}$ $\uparrow\{n\} = \{Y \in FC(\mathbb{N}) \mid n \in Y\}$ is an ultrafilter.
- $\{\text{cofinite subset of } \mathbb{N}\}$ is an ultrafilter.

EXERCISE: there are no other ultrafilters.

We are finally ready to prove

Stone's Representation Theorem for Boolean algebras (1936)

Every Bool. alg. B is isom. to a subalg. of $\mathcal{P}(X)$ for some set X.

Equivalently, there is an injective homomorphism $B \hookrightarrow \mathcal{P}(X)$ for some set X.

Proof

Set $X := \mathrm{Ult}(B)$.

Let us define a function

$$\eta_B : B \longrightarrow \mathcal{P}(\mathrm{Ult}(B))$$
$$b \longmapsto \{U \in \mathrm{Ult}(B) \mid b \in U\}.$$

and let us prove that it is

1) a Bool. hom. (EASY)

2) INJECTIVE (HARD)

1) We start by proving that η_B is a Bool. hom., i.e. η_B preserves $1, \wedge, \neg$

- $\eta_B(1) \overset{?}{=} \mathrm{Ult}(B)$

$$\eta_B(1) = \{U \in \mathrm{Ult}(B) \mid 1 \in U\} = \mathrm{Ult}(B)$$

since 1 belongs to any filter.

- $\eta_B(x \wedge y) \overset{?}{=} \eta_B(x) \cap \eta_B(y)$

$$\{U \in \mathrm{Ult}(B) \mid x \wedge y \in U\} \overset{?}{=} \{U \in \mathrm{Ult}(B) \mid x \in U\} \cap \{U \in \mathrm{Ult}(B) \mid y \in U\}$$

$\subseteq$) $x \quad y$, $x \wedge y \in U$ $\Rightarrow$ $x \in U$, $y \in U$ because filters are upwards closed.

$\supseteq$) $U \ni x$, $\ni y$ $\Rightarrow$ $x \wedge y \in U$ because any filter is closed under $\wedge$.

- $\eta_B(\neg x) \overset{?}{=} \mathrm{Ult}(B) \setminus \eta_B(x)$

$$\eta_B(\neg x) = \{U \in \mathrm{Ult}(B) \mid \neg x \in U\}$$

$$\eta_B(x) = \mathrm{Ult}(B) \setminus \{U \in \mathrm{Ult}(B) \mid x \in U\} = \{U \in \mathrm{Ult}(B) \mid x \notin U\}$$

$$\{U \in \mathrm{Ult}(B) \mid \neg x \in U\} \overset{?}{=} \{U \in \mathrm{Ult}(B) \mid x \notin U\}$$

for any ultrafilter, for every x, exactly one between x and $\neg x$ belongs to the ultrafilter

2) It remains to prove that η_B is injective.
Suppose $a, b \in B$ are such that $a \neq b$.
We shall prove that $\eta_B(a) \neq \eta_B(b)$.

$$\eta_B(a) = \{U \in \mathrm{Ult}(B) \mid a \in U\} \qquad \eta_B(b) = \{U \in \mathrm{Ult}(B) \mid b \in U\}$$

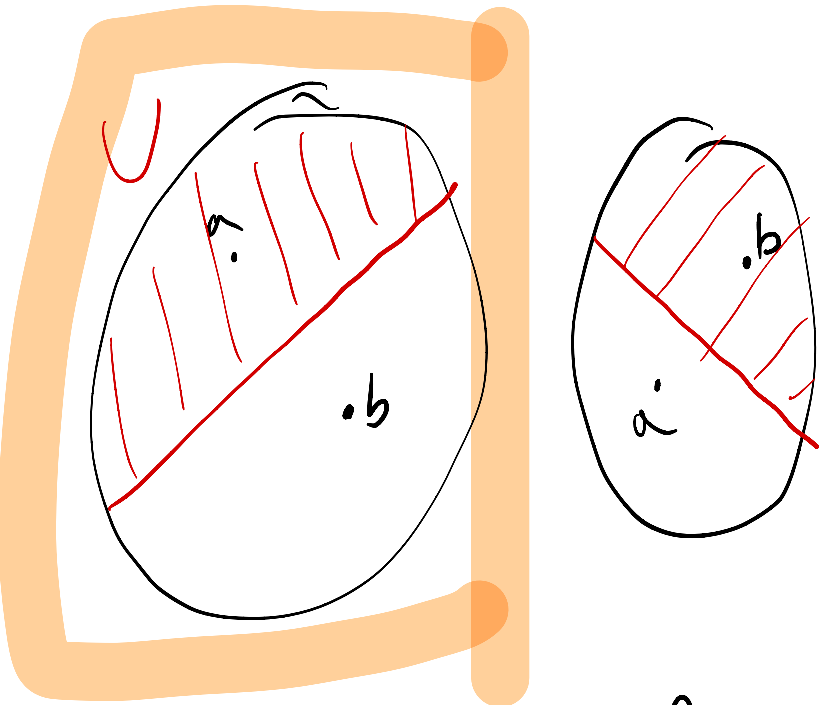
We look for an ultrafilter that contain one between a and b but not the other one.

$a \neq b \Rightarrow a \nleq b$ or $b \nleq a$.

These two conditions are perfectly symmetrical.

WLOG (without loss of generality), we can suppose

$$a \nleq b$$



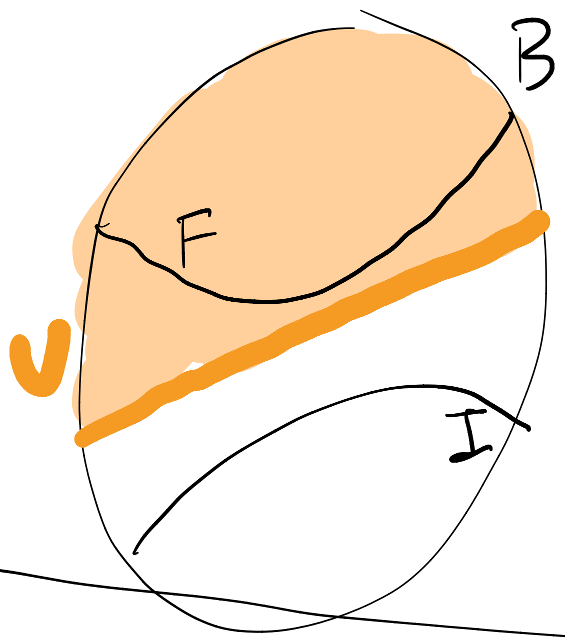
So, let us suppose $a \nleq b$.

We look for an ultrafilter U s.t. $a \in U$, $b \notin U$.

To do so, we prove

$\boxed{\text{Thm}}$ (Boolean Prime Ideal Theorem)

Let B be a Bool. alg., let F be a filter, let I be an ideal such that I and F are disjoint.
Then, there is an ultrafilter U s.t. $F \subseteq U$ and $U \cap I = \emptyset$.



In our case, we will use the Boolean Prime Ideal Thm by setting
$F := \uparrow a$

$I := \downarrow b$

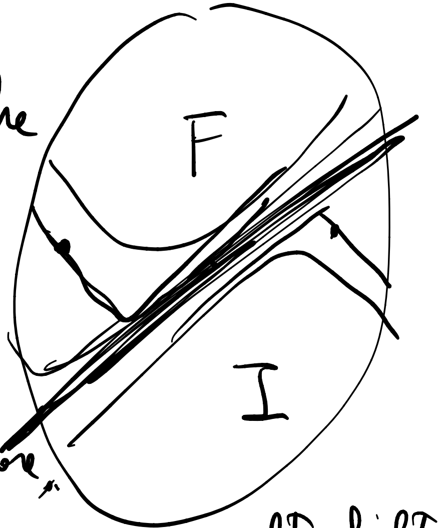
$F \cap I = \emptyset$ because otherwise there would be $x \in F \cap I = \uparrow a \cap \downarrow b$
but then $a \le x \le b$ which contradicts $a \not\le b$

Applying the Bool. Pr. Id. Thm to our setting will give an ultrafilter U s.t.
$\uparrow a \subseteq U$ and $\downarrow b \cap U = \emptyset$
$\Downarrow$ $\qquad$ $\Downarrow$
$a \in U$ $\qquad$ $b \notin U$.

We get the desired ultrafilter.
This will finish SRT's proof

Proof of the Boolean Prime Ideal Theorem

Idea: We progressively enlarge the filter F and the ideal I, still keeping the disjointness between the two, until they are so big that they cannot be enlarged anymore. Then we will prove that they are complimentary, and so the filter is an ultrafilter.



Zorn's lemma

Let P be a poset s.t.

- P is nonempty
- every nonempty $\boxed{\text{chain}}$ has an upper bound

(totally ordered subset)

Then, P has a maximal element.

Zorn's lemma is equiv. to the Ax. of Choice

$$P = \left\{ (G, J) \;\middle|\; \begin{array}{l} G \text{ filter} \\ J \text{ ideal} \\ F \subseteq G \\ I \subseteq J \\ G \cap J = \emptyset \end{array} \right\}$$

ordered by componentwise inclusion

$(G, J) \leq (G', J')$ iff $G \subseteq G'$, $J \subseteq J'$.

Next steps:

(A) Show that P satisfies the hypothesis of Zorn's lemma.

This will give a maximal element (U, K)

(B) Show that U and K are complimentary, and so U is an ultrafilter.

(A)

- $P \neq \emptyset$, because: $(F, I) \in P$,
- We show that every nonempty chain has an upper bound.

Let $S \subseteq P$ be a nonempty chain

$\forall (G,J), (G',J') \in S$

$\begin{cases} G \subseteq G' \\ J \subseteq J' \end{cases}$ or $\begin{cases} G' \subseteq G \\ J' \subseteq J \end{cases}$

Then consider $\left(\bigcup_{(G,J)\in S} G , \bigcup_{(G,J)\in S} J \right)$

We shall prove that it belongs to P, i.e.

- $\bigcup_{(G,J)\in S} G$ is a filter. LEMMA $\boxed{EX}$ A union of a nonempty totally ordered family of filters is a filter.
- $\bigcup_{(G,J)\in S} J$ is an ideal SIMILARLY FOR IDEALS.

- $F \subseteq \bigcup_{(G,J)\in S} G \rightsquigarrow S \neq \emptyset$, so there is a $(G_0,J_0)\in S \rightarrow$
  $F \subseteq G_0 \subseteq \bigcup_{(G,J)\in S} G$
- $I \subseteq \bigcup_{(G,J)\in S} J \rightsquigarrow$ Analogous.
- $\left(\bigcup_{(G,J)\in S} G\right) \cap \left(\bigcup_{(G,J)\in S} J\right) = \emptyset$.

$\hookrightarrow$ Let us prove it.
Suppose by contradict. there is $x \in \left(\bigcup_{(G,J)\in S} G\right) \cap \left(\bigcup_{(G,J)\in S} J\right)$

there are $(G_1, J_1)\in S$, $(G_2, J_2)\in S$

s.t. $x \in G_1$, $x \in J_2$

WLOG: $(G_1, J_1) \leq (G_2, J_2)$. Then $x \in G_1 \subseteq G_2$
S is tot. ord.

$\in J_2$ contradiction with $G_2 \cap J_2 = \emptyset$

We have proved it belongs to P.

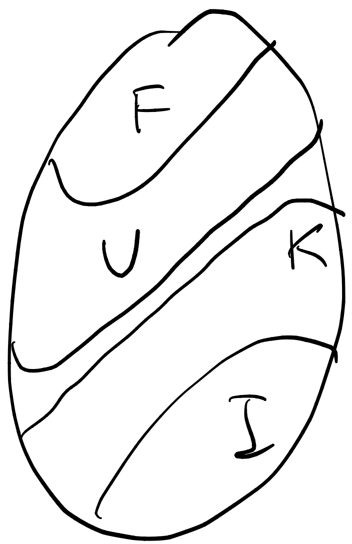
$$\left(\bigcup_{(G,J)\in S} G\right) \cap \left(\bigcup_{(G,J)\in S} J\right) \subseteq P$$

And it clearly is an upper bound for S.

So, we can apply Zorn's lemma.

$\Rightarrow$ P has a maximal element. (U,K)

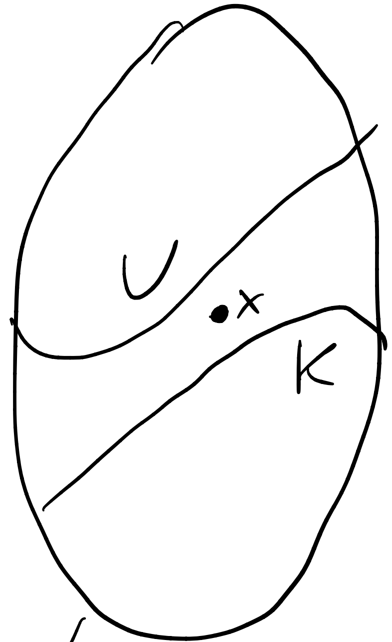
B



We want to prove that U is an ultrafilter;
it is enough to prove

$U \cup K = B$ (since we already know $U \cap K = \emptyset$ and that U is a filter and K is an ideal)

To prove $U \cup K = B$,
let us suppose, by way of contradiction, that
there is $x \in B \setminus (U \cup K)$,
and let us reach a contradiction

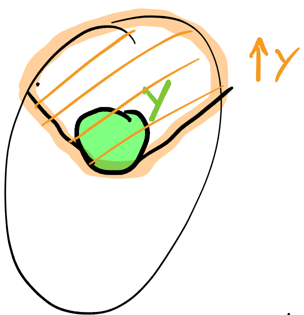


By maximality, we know that

- $\Big(\underbrace{\text{filter generated by } U \text{ and } x}_{U' := \uparrow\{u \wedge x \mid u \in U\}}, K\Big) \notin P$, i.e. $U' \cap K \neq \emptyset$
- $\Big(U, \underbrace{\text{ideal gen. by } K \text{ and } x}_{K' := \downarrow\{k \vee x \mid k \in K\}}\Big) \notin P$, i.e. $U \cap K' \neq \emptyset$

Notation In a poset X, for $Y \subseteq X$

$$\uparrow Y := \{x \in X \mid \exists y \in Y : y \leq x\}$$



Analogously, $\downarrow Y := \{x \in X \mid \exists y \in Y : x \leq y\}$

$U' \cap K \neq \emptyset \Rightarrow$ there is $u' \in U' \cap K$

$\geq$

$u \wedge x$ for some $u \in U$, because $u' \in U'$

Since $u' \in K$

$\geq$

$u \wedge x$,

we have $u \wedge x \in K$

$U \cap K' \neq \emptyset \Rightarrow$ there is $k' \in U \cap K'$

$\leq$

$k \vee x$ for some $k \in K$, because $k' \in K'$!

$$\left.\begin{array}{l} k' \in U \\ \leq \\ k \vee x \end{array}\right\} \Rightarrow k \vee x \in U$$

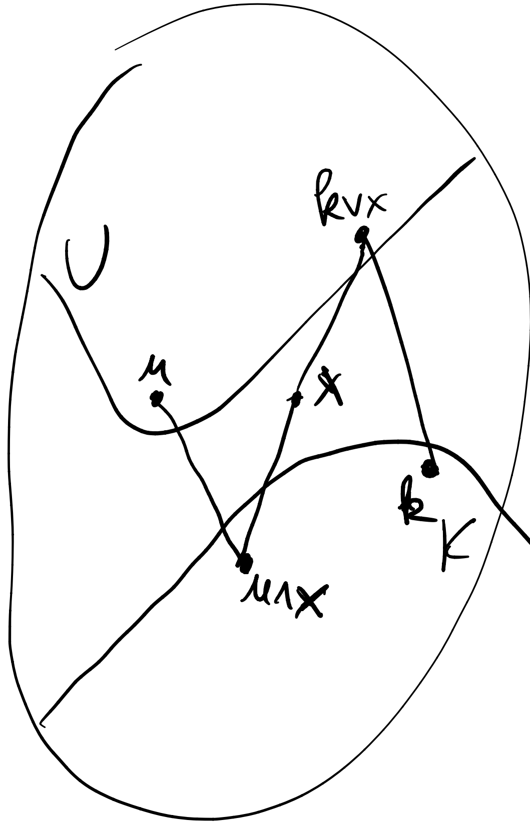
$u \in U,\ k \vee x \in U \Rightarrow u \wedge (k \vee x) \in U$

$\|$ by distributivity

$$\underbrace{(u \wedge k)}_{\in K} \vee \underbrace{(u \wedge x)}_{\in K} \in K$$

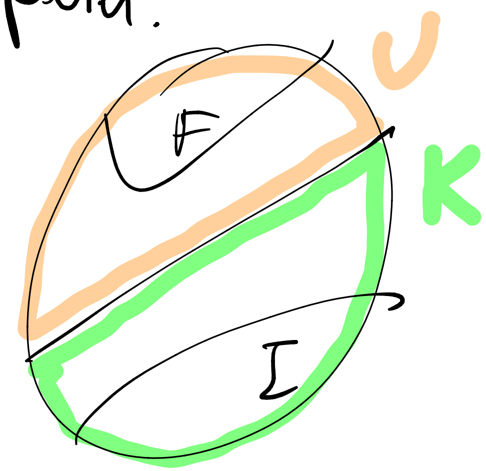
because $u \wedge k \leq k \in K$

$\Big\}$ contradicts $U \cap K = \emptyset$



In conclusion, $K \cup \mathcal{U} = B$, so U is an ultrafilter.

$$\left.\begin{array}{l} F \subseteq U, \quad K \cap U = \emptyset \\ I \subseteq K \end{array}\right\} \Rightarrow U \cap I \neq \emptyset$$



This finishes the proof of the Boolean Prime Ideal thm.

And so the proof of Stone's Repn. Theorem is also complete.

Stone's representation thm $\Leftrightarrow$ Bool. Prime Id. thm $\to$ cannot be proven in ZF.

The Boolean Prime Ideal theorem (or the Stone's representation theorem, which is equivalent, in ZF), is strictly weaker than the axiom of choice, meaning that it follows from the Ax. of choice, but not vice versa: the Axiom of Choice cannot be proved from the Boolean Prime Ideal

So, Stone's Repn. Thm and The Boolean Prime Ideal Theorem are a weak form of the axiom of choice.

The representation of a Boolean algebra as a subalgebra of a powerset is not unique.

E.g.:

$$: 2^{=\{0,1\}} \hookrightarrow \mathcal{P}(\{*\})$$
$$1 \longmapsto \{*\}$$
$$0 \longmapsto \emptyset$$

This is the "canonical" one: it is the one given by the proof of Stone's Repres. Theorem.

$$i : 2 \hookrightarrow \mathcal{P}(\{x,y\})$$
$$1 \longmapsto \{x,y\}$$
$$0 \longmapsto \emptyset$$

This is not "canonical"; it is not "separated", i.e. the image of i "does not separate x and y".

"Separation" is not the only property that the canonical representation satisfies: there is also another one: "COMPACTNESS"

$$FC(\mathbb{N}) \hookrightarrow \mathcal{P}(\mathbb{N})$$
$$X \longmapsto X$$

$$j : FC(\mathbb{N}) \hookrightarrow \mathcal{P}(\mathbb{N} \cup \{\infty\})$$
$$X \longmapsto \begin{cases} X & \text{if } X \text{ is finite} \\ X \cup \{\infty\} & \text{if } X \text{ is cofinite} \end{cases}$$

This is the canonical one! It is "compact": every cover of $\mathbb{N}\cup\{\infty\}$ by elements in the image of j has a finite subcover.

SPOILER: SEPARATION and COMPACTNESS CHARACTERIZE CANONICAL REPRESENTATIONS

A natural setting in which to talk about separation and compactness is the setting of topology.
This will lead to the notion of a <u>Stone</u> space.