3rd Lecture
2026.02.18

LAST TIME:

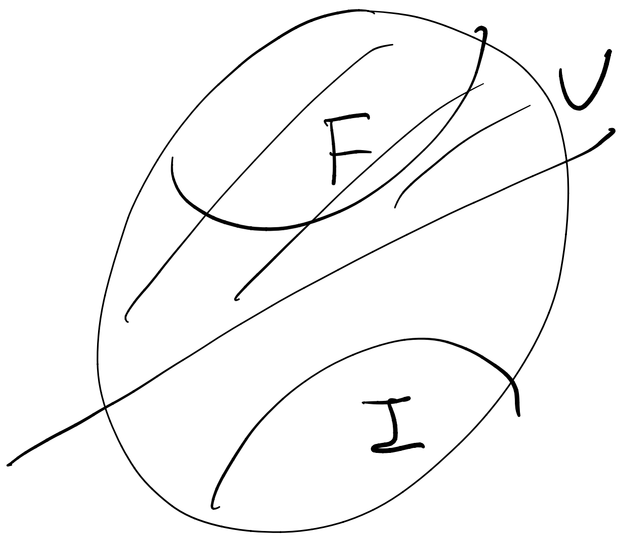
Stone's Repr. Thm

$$\eta_B \quad B \hookrightarrow \mathcal{P}(\mathrm{Ult}(B))$$
$$b \longmapsto \{U \in \mathrm{Ult}(B) \mid b \in U\}$$

HOM (EASY)

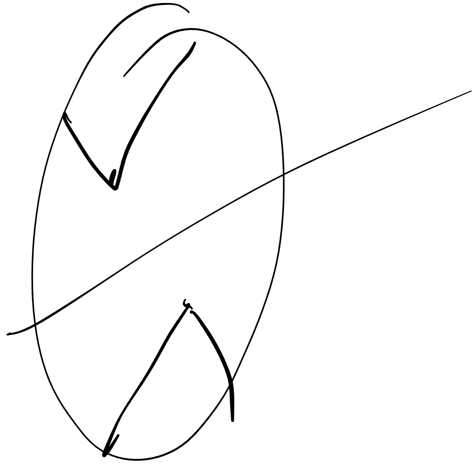
INJ (HARD)

Boolean Prime Ideal Theorem



Separation by Ultrafilter Theorem

For $a, b \in B$ with $a \neq b$, there is an ultrafilter containing exactly one between a and b.



From a set-theoretic perspective

→ Every Bool. subalg. of a powerset is a Bool. algebra (SOUNDNESS)

→ Stone's RT says that the list of axioms is complete (COMPLETENESS)

LOGICAL.

• For any theory $T \subseteq \mathrm{Form}(\mathcal{L})$ in a propositional language $\mathcal{L}$

$$\mathrm{Form}(\mathcal{L}) / \equiv_T$$

$\varphi \equiv_T \psi$ means that in every model $v : \mathcal{L} \to \{0,1\}$ of T (i.e.: $\forall \sigma \in T \ \bar{v}(\sigma) = 1$)

$\bar{v}(\varphi) = \bar{v}(\psi)$.

LINDEMBAUM-TARSKI ALGEBRA ASSOCIATED TO T

$\mathrm{Form}(\mathcal{L})/_{\equiv_T}$ is a Bool. alg.

(SOUNDNESS)

$$\begin{tikzcd} \mathcal{L} \arrow[r, hook] \arrow[dr, "v"'] & \mathrm{Form}(\mathcal{L}) \arrow[d, dashed, "\bar{v}"] \\ & \{0,1\} \end{tikzcd}$$

Conversely, <u>any</u> Bool. alg. is isomorphic to $\mathrm{Form}(\mathcal{L})/_{\equiv_T}$ for some $T, \mathcal{L}$.

(COMPLETENESS)

<u>Before proving it, EXAMPLES</u>

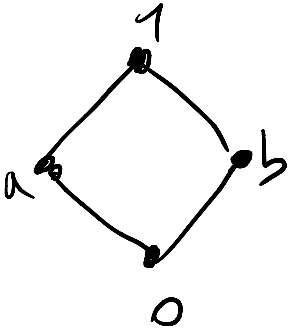
$2 \cong \mathrm{Form}(\mathcal{L})/_{\equiv_T}$, with $\mathcal{L} = \emptyset$, $T = \emptyset$

$\mathrm{Form}(\emptyset) = \{0, 1, 0 \vee 0, 0 \vee 1, \neg(0 \wedge 1), \dots\}$

$$\begin{tikzcd} \emptyset \arrow[r, hook] \arrow[dr, "v"'] & \mathrm{Form}(\mathcal{L}) \arrow[d, dashed, "\bar{v}"] \\ & \{0,1\} \end{tikzcd}$$

$\mathrm{Form}(\mathcal{L}) \ni 0 \mapsto 0$, $0 \vee 0 \mapsto 0$, $1 \mapsto 1$

$1 \not\equiv_{\emptyset} 0$



$\mathcal{L} = \{p\}$

$T = \emptyset$

$Fm(\mathcal{L}) = \{\underbrace{0, p, \neg p, 1}, p \vee p, \dots\}$

The Trivial Bool. alg

$\mathcal{L} =$ any

$T = Fm(\mathcal{L})$

$\mathcal{L} = \emptyset$

$T = \{0\}$

There are no model of T.

$$\begin{tikzcd} \mathcal{L}_0 \arrow[r] \arrow[dr, "v"'] & Fm(\mathcal{L}_0) \arrow[d, dashed] \\ & \{0,1\} \end{tikzcd} \qquad \begin{array}{c} 0 \in T \\ \downarrow \\ 0 \end{array} \qquad \lightning$$

And so any two formulas ~~coincide~~

$\equiv_T$

$FC(\mathbb{N}) = \{$finite/cofinite subset of $\mathbb{N}\}$

$x \leftrightarrow y$
$= (x \to y) \wedge (y \to x)$
$= (\neg x \vee y) \wedge (\neg y \vee x)$

0 1 2 3

$\mathcal{L} = \{p_0, p_1, \ldots\}$

for $i \neq j$ $\quad \neg(p_i \wedge p_j)$

$\mathbb{N}$

$T = \{\neg(p_i \wedge p_j) \mid i \neq j\}$

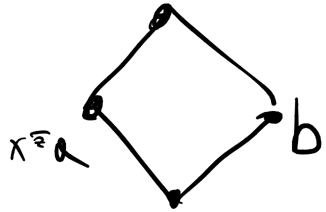
Let B be a Bool. alg. How to find $\mathcal{L}, T$ s.t.

$B \simeq Form(\mathcal{L})/\equiv_T$?

$\mathcal{L} := B$

T:

- for any $x \in B$, $\neg_B x \,\dot{\leftrightarrow}\, \dot{\neg} x$
- for any $x, y \in B$, $(x \wedge_B y) \,\dot{\leftrightarrow}\, (x \,\dot{\wedge}\, y)$
- $1_B \,\dot{\leftrightarrow}\, \dot{1}$



$\neg_B : B \to B$

$\dot{\neg}$ for the syntax in Form($\mathcal{L}$)

$\dot{\neg} x$

$\neg_B x$

CLAIM: $B \simeq \mathrm{Form}(\mathcal{L})/_{\equiv_T}$

PROOF

$$\begin{tikzcd} B \arrow[r, hook] \arrow[dr, "id"'] & \mathrm{Form}(B) \arrow[r, two heads, "\pi"] \arrow[d, dashed, "\exists! \widehat{id}"] & \mathrm{Form}(B)/_{\equiv_T} \arrow[dl, "\sim"] \\ & B & \end{tikzcd}$$

$$\begin{tikzcd} \{\vee,\wedge,\neg,0,1\}\text{-}\mathrm{Alg} \arrow[r, bend right, phantom, "\perp"] & \mathrm{Set} \arrow[l, bend right, "F"'] \end{tikzcd}$$

$$\mathrm{Form}(\mathcal{L}) \longleftarrow\!\shortmid \mathcal{L}$$

$$\begin{tikzcd} \mathcal{L} \arrow[r, hook] \arrow[dr, "\forall f"'] & \mathrm{Form}(\mathcal{L}) \arrow[d, dashed, "\exists! \bar{f} \text{ homo}"] \\ & B \in \Sigma\text{-}\mathrm{Alg} \end{tikzcd}$$

FUNCTION

$$\begin{tikzcd} \mathcal{L} \arrow[r, hook] \arrow[dr, "v"'] & \mathrm{Form}(\mathcal{L}) \arrow[d, dashed, "\bar{v}"] \\ & 2 \end{tikzcd}$$

LEMMA

Given a diagram in Σ-Alg

$$\begin{tikzcd} A \arrow[r, two heads, "g"] \arrow[d, "f"'] & C \\ B & \end{tikzcd}$$

with f and g surj.

if, for all $a, a' \in A$

we have $f(a) = f(a')$

$\Updownarrow$

$g(a) = g(a')$

Then there is an iso $B \simeq C$ making the diagram commute

$$\begin{tikzcd} A \arrow[r, two heads, "g"] \arrow[d, two heads, "f"'] & C \\ B \arrow[ur, "\simeq"'] & \end{tikzcd}$$

IN VIEW OF THIS LEMMA, TO PROVE $B \simeq \mathrm{Form}(B)/_{\equiv_T}$,

it is enough to prove : For all $\varphi, \psi \in \mathrm{Form}(B)$

$$\overline{id}(\varphi) = \overline{id}(\psi) \Longleftrightarrow [\varphi]_{\equiv_T} = [\psi]_{\equiv_T}$$

$$\Updownarrow$$

$$\varphi \equiv_T \psi$$

$$\Updownarrow$$

$$\boxed{\text{for all } v : B \to 2 \text{ models of } T \quad \bar{v}(\varphi) = \bar{v}(\psi)}$$

$$\Updownarrow$$

EXERCISE: For every $v : B \to 2$. TFAE:

1) v is a model of T

2) v is a homomorphism.

for all homom. $v : B \to 2$, $\bar{v}(\varphi) = \bar{v}(\psi)$.

$\Rightarrow$) EASY. EXERCISE

$\Leftarrow$) ("HARD") Let us prove the contrapositive

$$\underbrace{\overline{id}(\varphi)}_{\in\, B} \neq \underbrace{\overline{id}(\psi)}_{\in\, B}$$

By the "Separation by ultrafilter theorem", there is an ultrafilter containing one between $\overline{id}(\varphi)$ and $\overline{id}(\psi)$ but not the other one, i.e. there is a homom. $v : B \to 2$ s.t. $v(\overline{id}(\varphi)) \neq v(\overline{id}(\psi))$

CLAIM: $\overline{v}(\varphi) \neq \overline{v}(\psi)$.

By uniqueness in the univ. prop. of the unit, $\overline{v} = v \circ \overline{id}$. This finishes the claim.

$2 \hookrightarrow \mathcal{P}(\{*\})$

SEPARATED

$i : 2 \hookrightarrow \mathcal{P}(\{x,y\})$

$0 \longmapsto \emptyset$

$1 \longmapsto \{x,y\}$

NOT SEPARATED

$FC(\mathbb{N}) \hookrightarrow \mathcal{P}(\mathbb{N})$

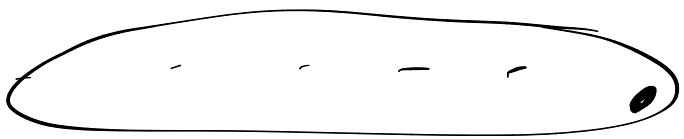
$Z \longmapsto Z$

$$\mathbb{N} = \bigcup_{n\in\mathbb{N}} \{n\}$$

IS NOT COMPACT

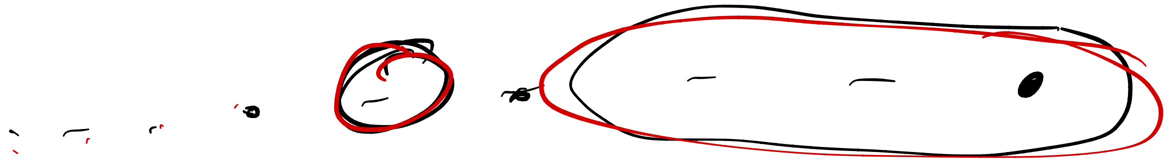
$j : FC(\mathbb{N}) \hookrightarrow \mathcal{P}(\mathbb{N}\cup\{\infty\})$

$$Z \longmapsto \begin{cases} Z & \text{if } Z \text{ is finite} \\ Z\cup\{\infty\} & \text{if } Z \text{ is cofinite.} \end{cases}$$



COMPACT

→ Every cover of $\mathbb{N}\cup\{\infty\}$ by elements in the image of j has a finite subcover.

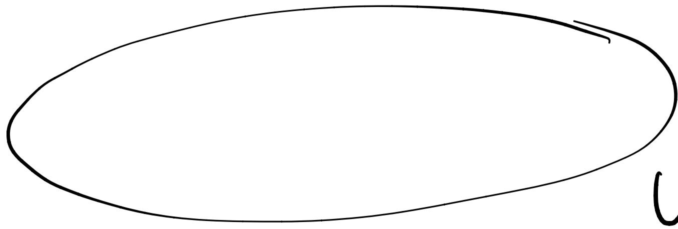


SPOILER: SEPARATION + COMPACTNESS CHARACTERIZES CANONICAL REPRESENTATIONS.

$$\eta_B : B \hookrightarrow \mathcal{P}(Ult(B))$$

LEMMA (Canonical rep. is "separated")

Let B be a Bool. Alg.
Let $U, U' \in Ult(B)$ such that $U \neq U'$.
Then, there is $b \in B$ s.t.
$\eta_B(b)$ contains exactly one between U and U'.
$\ni$



$Ult(B)$

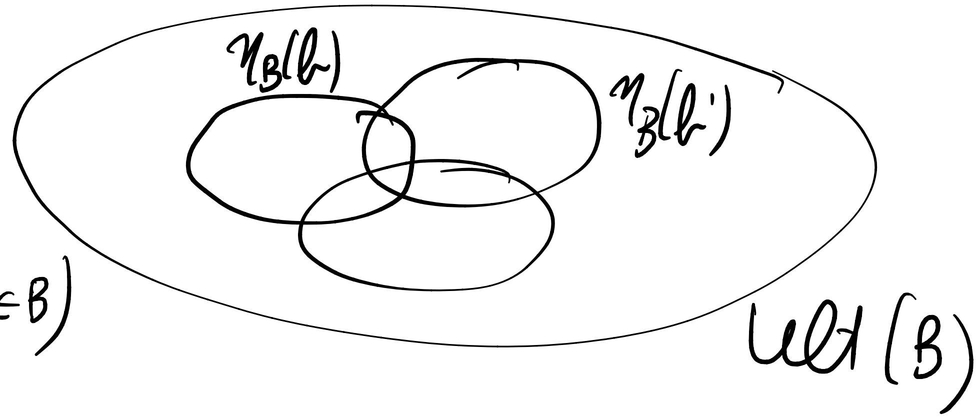
PROOF $U \neq U' \Rightarrow \exists b \in B$ s.t. b belongs to exactly one of the two.

$b \in U$ $\Updownarrow$ $U \in \eta_B(b)$

LEMMA (Canonical repr. in "compact")

Let B be a Bool alg.

Every cover of $Ult(B)$ by elements of the form $\eta_B(b)$ $(b \in B)$ has a finite subcover



PROOF

$Ult(B) = \bigcup_{i \in I} \eta_B(b_i) \rightsquigarrow$ For every ultrafilter V, $\exists i \in I$: $V \in \eta_B(b_i)$
$b_i \in V$

Consider the ideal generated by $\{b_i\}$: $J := \downarrow\{b_{i_1} \vee \dots \vee b_{i_n} \mid i_1, \dots, i_n \in I\}$

CLAIM: $1 \in J$.

Proof of claim: BWOC, suppose not: $1 \notin J$.

$\{1\}$ is a filter, disjoint from J.

$\Rightarrow$ By the Bool. Prim Id. Thm, there is $U \in Ult(B)$ s.t. $\{1\} \subseteq U$, $U \cap J = \emptyset$.

$\Rightarrow$ By hypoth. $\exists i \in I$ s.t. $\underset{\underset{J}{\cap}}{b_i} \in V$ $\Downarrow$ $V \cap J = \emptyset$

This proves the claim: $1 \in J$.

$1 \in I = \downarrow \{ b_{i_1} \vee \dots \vee b_{i_n} \mid i_1 \dots i_n \in I \}$

$\eta_B(1) \leq \eta_B(b_{i_1}) \vee \dots \vee \eta_B(b_{i_n})$

$\quad \shortparallel \qquad =$

$\mathrm{Ult}(B)$

$B \xhookrightarrow{\sim} \mathcal{P}(X) \qquad \tau \subseteq \mathcal{P}(X)$

Let's Topologize.

Stone spaces $\longleftrightarrow$ Bool. Alg.

Given a Top. space X, we have a Bool. Alg.

$\{\text{clopens of } X\}$

Clopen = closed and open

EXAMPLE: In a discrete space: every subset is clopen

In $[0,1]$ $\quad [0,1]$ $\emptyset$.

In $[0,1] \cup [2,3]$: $\emptyset$, $[0,1]$, $[2,3]$, $[0,1] \cup [2,3]$

$$0 \;\cdots\; \tfrac{1}{4} \;\; \tfrac{1}{3} \;\; \tfrac{1}{2} \;\cdots\; 1 \quad \subseteq \mathbb{R}$$

The clopens: for every finite subset Z of $\mathbb{N} \setminus \{0\}$,

$\{\frac{1}{n} \mid n \in Z\}$ is a clopen,

and also its complement.

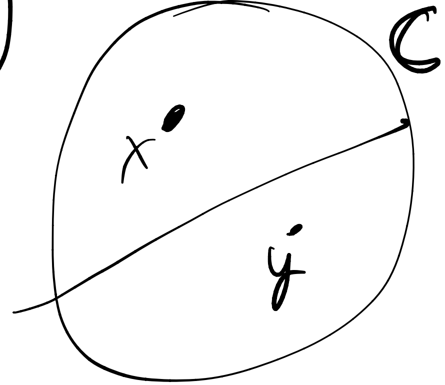
EX: these are the only clopens.

$$\mathrm{Clop}(X) \simeq FC(\mathbb{N})$$

Hausdorff. $\nearrow$

Def A Stone space is a topological space X s.t.

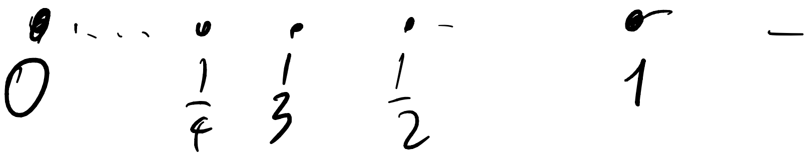
1. X is Totally separated, i.e. "clopens separate the points", i.e. for all $x, y \in X$ with $x \neq y$ there is a clopen C s.t. $x \in C$, $y \notin C$.
2. X is compact (open covers have finite subcover)



EQUIV: compact Hausdorff totally disconnected

" " 0-dimensional

EXAMPLE: Finite discrete space. (Every finite Stone space is discrete)



one-point Alexandroff compactif. of $\mathbb{N}$,

Open: → any set not containing ∞
→ any cofinite set containing ∞

Clopens: → finite sets not containing ∞
→ cofinite sets containing ∞.

NON-EXAMPLES → $\{x, y\}$ indiscrete (Opens: $\emptyset$, $\{x, y\}$)

→ $[0,1]$

→ Any infinite discrete space.

$$\text{Stone spaces} \longleftrightarrow \text{Bool. Alg}$$

$$X \longmapsto \mathrm{Clop}(X)$$

$$\mathrm{Ult}(B) \longleftarrow\!\shortmid \; B$$

We topologize $\mathrm{Ult}(B)$, making it a Stone space, with the <u>Stone</u> Topology.

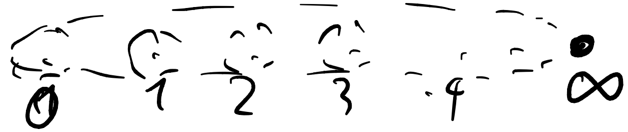
The family $\{\eta_B(b) \mid b \in B\}$ is closed under finite intersections, so it forms a basis for a topology.

<u>Stone</u> topology := the topology generated by the sets of the form $\eta_B(b)$, $b \in B$

$\parallel$

set of arbitrary unions of elements of the form $\eta_B(b)$.

$FC(\mathbb{N})$



<u>LEMMA</u>

$Ult(B)$ with the Stone top., is compact

PROOF: to check compactness it is enough to check it on a basis.

THIS IS "the canonical representation is compact".

LEMMA ⊛

The clopens of $Ult(B)$ are precisely the sets of the form $\eta_B(b)$, for $b \in B$

PROOF: EXERCISE

LEMMA: $Ult(B)$ is Totally separated

PROOF This is "The canonical reps. is separated.

THM.
$$\begin{aligned} \eta_B : B &\longrightarrow Clop(Ult(B)) \\ b &\longmapsto \{U \in Ult(B) \mid b \in U\} \end{aligned}$$
is an iso of Bool. Alg.

$$\eta_B : B \hookrightarrow P(Ult(B))$$

By $\otimes$

$$\begin{tikzcd} \text{Stone Sp} \arrow[r, shift left, "Clop"] & \text{Bool. Alg} \arrow[l, shift left, "Ult"] \end{tikzcd}$$

THM

Given a Stone space X,

$$\varepsilon_X : X \longrightarrow \mathrm{Ult}(\mathrm{Clop}(X))$$

$$x \longmapsto \{A \in \mathrm{Clop}(X) \mid x \in A\}$$

is a homeomorphism.

PROOF

- CONT. $\Leftarrow$ EASY.
- INJ. $\Leftarrow$ X TOTALLY SEPARATED
- SURJ. $\Leftarrow$ X COMPACT

(Any bij cont. map between compact Hausdorff spaces is a homeomorphism)

BECAUSE in comp. Hausd.

CLOSED $\Longleftrightarrow$ COMPACT.

and so every continuous map is closed