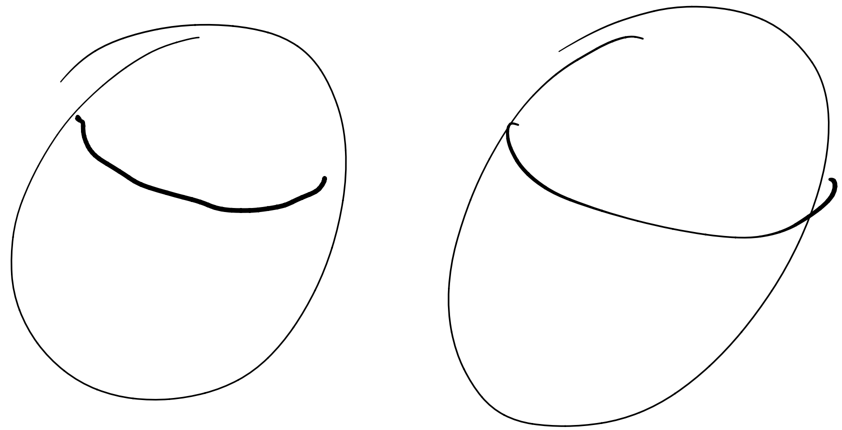
• ERRATA CORRIGE: 1 Stone $\to$ Set

preserves finite coproducts

INIT: $\emptyset$

BIN. COPRODUCT: $X+Y$ is the disjoint union with the disjoint union topology.



2 Stone $\to$ Set does not preserve coequalizers

(see example few pages below).

BA is a variety of algebras (= eq. definable).

$\Rightarrow$ complete and cocomplete.

Therefore, Stone is complete and cocomplete

Stone $\to$ Set does not preserve arbitrary colimits

$$1 + 1 + \dots$$

So Stone $\to$ Set is not a left adjoint.

Does Stone $\to$ Set have a left adjoint?

$$F : \mathrm{Set} \to \mathrm{Stone}?$$

If F exists then it should satisfy

$$F(\underset{\uparrow \atop \text{Set}}{X}) = F\Big(\sum_{x\in X} \{*\}\Big) = \sum_{x\in X} F(\{*\})$$

$$\hom_{\text{Stone}}\big(\underbrace{F(\{*\})}, \underset{\uparrow \atop \text{Stone}}{Y}\big) \simeq \hom_{\text{Set}}(\{*\}, UY) \simeq UY$$

We can take $F(\{*\}) = \{*\}$

$\text{Stone} \to \text{Set}$ is nat. isom. to $\hom_{\text{Stone}}(\{*\}, -)$

$$F(X) := \sum_{x\in X} \{*\}$$

$\sum_{n\in\mathbb{N}} \{*\}$ in Stone ?

in the dual of $\underbrace{\prod_{n\in\mathbb{N}} 2}_{\simeq P(\mathbb{N})}$ in Bool.alg.

$$\sum_{n\in\mathbb{N}} \{*\} \simeq \mathrm{Ult}(P(\mathbb{N}))$$

THM: If C is a cocomplete category, a faithful functor

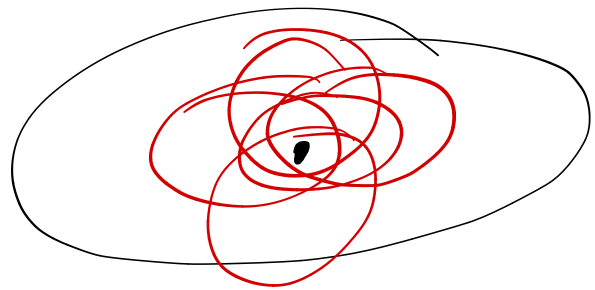
$$C \to \mathrm{Set}$$

has a left adjoint if and only if it is representable $\Leftarrow F(X) := \sum_{x\in X} Z$

$$\sum_{x\in X} \{*\} \simeq \mathrm{Ult}(P(X))$$

$$\mathrm{Ult}(P(X)) \leftarrow\!\shortmid X$$

$$\mathrm{Stone} \underset{\perp}{\rightleftarrows} \mathrm{Set} \ni X$$



$$\mathrm{Ult}(P(X)) \leftarrow\!\shortmid X$$

$$\mathrm{CompHaus} \underset{\perp}{\rightleftarrows} \mathrm{Set} \ni X$$

$$\mathrm{Set} \ni X \longrightarrow \mathrm{Ult}(P(X)) \ni U$$

$$x \longmapsto \{A \in P(X) \mid x \in A\}$$

$$X \xrightarrow[\text{FUNCTION}]{f} Y$$

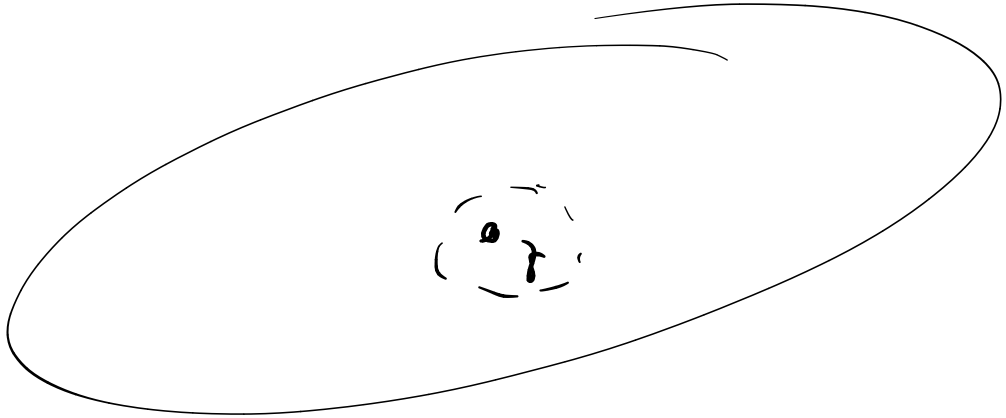
$\exists!\ \bar f$ cont. function.

$$Y \in \mathrm{CompHaus}$$

$$U \longmapsto f(\lim U)$$

$\lim \mathcal{U}$ is the unique element $z \in X$ s.t. $\mathcal{U}$ converges to z.

i.e. every neighbourhood of z belongs to $\mathcal{U}$



We use it to prove that Stone $\to$ Set does not preserve coequalizers.

CLAIM: for every comp. Hausd. space X there is a Stone space Y and a surject. cont. function

$$Y \twoheadrightarrow X$$

$$
\begin{tikzcd}
\overbrace{[0,1]}^{\in \mathrm{Set}} \arrow[r] \arrow[dr, "\mathrm{id}"'] & \mathrm{Ult}(P([0,1])) \in \mathrm{Stone} \arrow[d, dashed, "\exists!\, f \text{ cont. function}"] \\
& \underbrace{[0,1]}_{\in \mathrm{CompHaus}}
\end{tikzcd}
$$

this is surj. because id is surjective

Let

$$
\underset{\in \mathrm{Stone}}{Y} \overset{f}{\twoheadrightarrow} [0,1]
$$

any such map

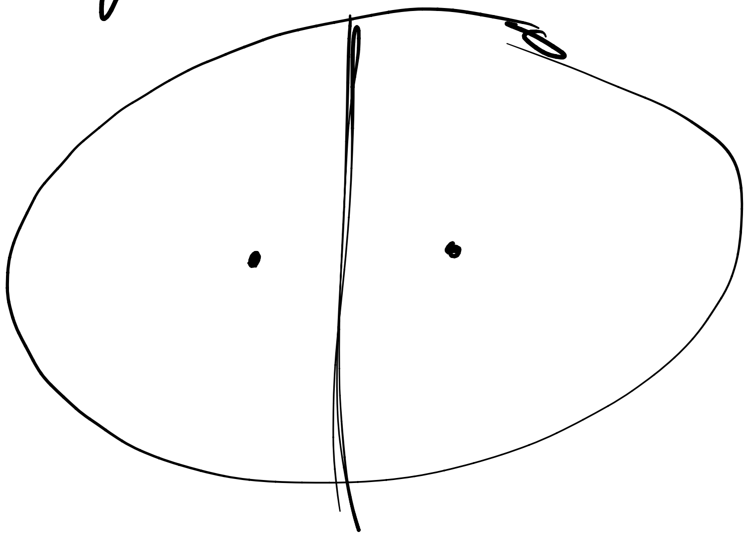
$$\begin{tikzcd}
Y\times Y & \mathrm{Stone} & & \\
\{(y_1,y_2)\in Y \mid f(y_1)=f(y_2)\} \arrow[u, phantom, "\subseteq" rotate=90] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, phantom, "\in" rotate=-90] & Y \arrow[u, phantom, "\in" rotate=90] \arrow[r, "f"] \arrow[dr] & {[0,1]} \arrow[d, dashed, "\exists! g"] & \text{connected} \\
\mathrm{Stone} & & Z \in \mathrm{Stone} &
\end{tikzcd}$$

one can show that this is surj.

CLAIM: Z is the singleton

The image $g([0,1])$ is connected

The only connected subsets are the singletons



LIMITS AND COLIMITS IN STONE AND BA

Both are complete and cocomplete.

In Both cases, limits are computed as in Set.

for BA this is because this hold for any variety of algebras.

because the forgetful functor

$$BA \longrightarrow Set$$

is isom.

$$\hom(\mathrm{Free}_{BA}(1), -)$$

THM (inj $\Leftrightarrow$ surj, surj $\Leftrightarrow$ inj)

Let X, Y be Stone spaces, $f: X \to Y$ a cont. function

1. f is inj $\iff$ $f^{-1}[-]: Clop(Y) \to Clop(X)$ is surjective
2. f is SURJ $\iff$ $f^{-1}[-]: Clop(Y) \to Clop(X)$ is injective

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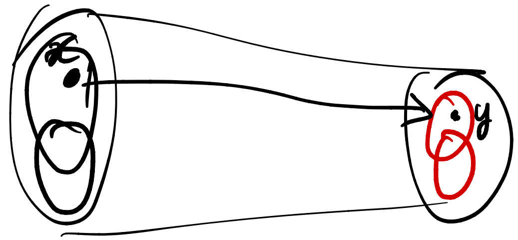
PROOF 1 already done.

2 $\Rightarrow$) Suppose f surj.



Let $A, B \in Clop(Y)$ s.t. $A \neq B$

CLAIM: $f^{-1}[A] \neq f^{-1}[B]$.

There is y belonging to A and not B (or vice versa)

By surj., there is x s.t. $f(x) = y$.

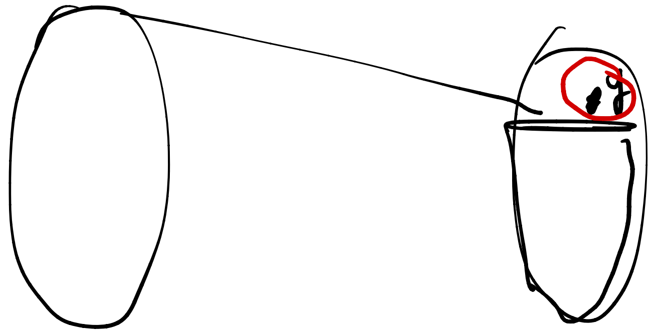
$$x \in f^{-1}[A] \setminus f^{-1}[B],$$ proving the claim.

$\Leftarrow)$ Suppose f is not surj. Let us prove $f^{-1}[-]$ is not inj.

There is $y \in Y \setminus f[X]$

$\Rightarrow$ There is a clopen

C s.t. $y \in C$



$C \cap f[X]$.

$\Rightarrow f^{-1}[C] = \emptyset \neq f^{-1}[\emptyset]$

$f^{-1}[-]$ is not inj, because $C \neq \emptyset$.

y

IDEA:

INJ. CONT. FUNCT.

SURJ. HOM.

$$\mathrm{Form}(\mathcal{L})/_{\equiv_T} \twoheadrightarrow \mathrm{Form}(\mathcal{L})/_{\equiv_{T'}}$$

$T \subseteq T'$

extend the theory

$$\mathrm{Mod}(T') \hookrightarrow \mathrm{Mod}(T)$$

less models.

$$B \hookrightarrow \mathrm{Form}(\mathcal{L})/_{\equiv_T}$$

"fewer propositions"

$$\mathrm{Mod}(T) \twoheadrightarrow Y$$

certain models become equivalent.

$A \to A+B$ inj?

$X \times Y \xrightarrow{\pi_X} X$ is surj?

$(x, \quad) \mapsto x$

Y is nonempty π_X is surj.

If $Y = \emptyset$ it is surj iff $X = \emptyset$.

	$Y=\emptyset$	$Y\neq\emptyset$
$X=\emptyset$	SURJ	SURJ
$X\neq\emptyset$	NOT SURJ	SURJ

$A \to A+B$ inj?

	$B=\{*\}$	$B\neq\{*\}$
$A=\{*\}$	INJ	INJ
$A\neq\{*\}$	NOT INJ	INJ

LOGICAL

$T_A \qquad \varphi \in \mathcal{L}_A$

INJ: $\forall \varphi \in \mathrm{Form}(\mathcal{L}_A) \left(T_A \not\vdash \varphi \Rightarrow T_A \sqcup T_B \not\vdash \varphi \right)$

Stone	Bool. Alg.
EPI = SURJ	MONO $\overset{?}{=}$ INJ
REG. EPI = SURJ.	REG. MONO $\overset{?}{=}$ INJ
MONO = INJ.	EPI $\overset{?}{=}$ SURJ
REG. MONO = INJ.	REG. EPI = SURJ.

Both in Stone and BA.

Epi = extr. epi = reg. epi = SURJ.

Mono = extr. mono = reg. mono = INJ.

THM. In every variety,
mono = inj.
REG. EPIS = SURJ.

EPI of Bool. alg.

CLAIM: EPI = SURJ.

$$\text{EPI} \Rightarrow \text{SURJ.}$$

$$\text{SURJ} \Rightarrow \text{EPI} \quad \checkmark$$

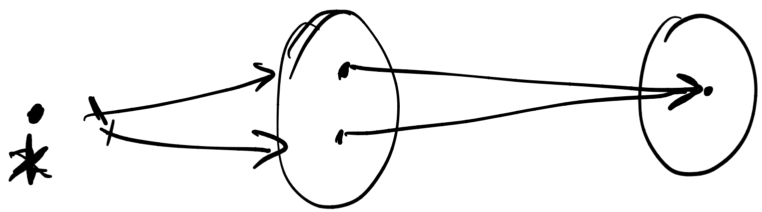
$$\overset{?}{\Rightarrow} \text{not epi}$$

$$A \xrightarrow{\text{not surj.}} B \rightrightarrows C$$

WE USE DUALITY:

We prove in Stone MONO $\Rightarrow$ INJ.

(eq. not inj. implies not mono)



in Stone MONO $\Rightarrow$ INJ.

Hence, in BA: EPI $\Rightarrow$ SURJ.

IN LOGICAL TERMS, surjectivity of epimorphisms corresponds to "Beth definability".

REG MONO of Bool. alg.

We prove REG EPIS of Stone spaces are surj.

SURJ $\Rightarrow$ REG. EPI.

$f[f^{-1}[\bar{g}^{-1}[C]]] \rightarrow g^{-1}[C] \subseteq X$ closed

$\Downarrow$

$f[g^{-1}[C]]$ is closed

$$\begin{tikzcd}
\{(x,y)\in X\times X \mid f(x)=f(y)\} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \arrow[r, two heads, "f"] \arrow[dr, dashed, "g"'] & Y \arrow[d, dashed, "\bar{g}"] \\
& & Z
\end{tikzcd}$$

$\cap$

$X\times X$

$Z \supseteq C$ closed

$\bar{g}^{-1}[C]$ closed?

$\bar{g}$ function

so $\bar{g}$ is continuous.

So, f is $\mathrm{coeq}(\pi_1, \pi_2)$. So f is reg. epi.

AMALGAMATION (logical Terms: INTERPOLATION)

$$
\begin{tikzcd}
 & B \arrow[dr, dashed, hook] & \\
A \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & & D \\
 & C \arrow[ur, dashed, hook] &
\end{tikzcd}
$$

i_1, i_2 INJ. of BA,

$$
\begin{tikzcd}
 & Y \arrow[dr, "f"] & \\
\{(y,z)\in Y\times Z \mid f(y)=g(z)\} \arrow[ur, dashed] \arrow[dr, dashed] & & X \\
 & Z \arrow[ur, "g"'] &
\end{tikzcd}
$$

(y,z)

$y \mapsto x = f(y) = g(z)$

$z \mapsto x$

How many formulas, up to logical equivalence, are there with n propositional symbols?

$n=0$ $\quad [0], [1]$

$n=1$

$\{p\}$ $\quad [0], [p], [\neg p], [1]$

I.e.: $\left|\mathrm{Form}(\mathcal{L})/_{\equiv}\right| = ?$

CLAIM: $\mathrm{Form}(\mathcal{L})/_{\equiv} \cong \mathrm{Free}_{BA}(\mathcal{L})$

$$
\begin{tikzcd}
\mathcal{L} \arrow[r] \arrow[dr, bend right, "v"'] & \mathrm{Free}_{BA}(\mathcal{L}) \arrow[d, dashed, "\exists!\, w \text{ Bool. hom.}"] \\
& A
\end{tikzcd}
$$

$A \in BA$

φ, ψ

$$\begin{tikzcd}
\mathcal{L}_0 \arrow[r] \arrow[dr, "v"'] & \mathrm{Form}(\mathcal{L}_0) \arrow[r, two heads, "\pi"] \arrow[d, dashed, "\bar{v}"] & \mathrm{Form}(\mathcal{L}_0)/\equiv \arrow[dl, dashed, "? \ \exists! w"] \\
 & B \arrow[d, "g"'] & \\
 & 2 &
\end{tikzcd}$$

$\{\vee, \wedge, \neg, 0, 1\}$-

$\uparrow$ BA

BWOC $\bar{v}(\varphi) \neq \bar{v}(\psi)$

Let $\varphi, \psi \in \mathrm{Form}(\mathcal{L}_0)$ s.t. $\pi(\varphi) = \pi(\psi)$, i.e. for every $w : \mathcal{L}_0 \to 2$

$$\bar{w}(\varphi) = \bar{w}(\psi)$$

$$\bar{w} : \mathrm{Form}(\mathcal{L}_0) \to 2$$

Does it follow that $\bar{v}(\varphi) = \bar{v}(\psi)$

if not, there is a homo $B \xrightarrow{g} 2$ s.t. $g(\bar{v}(\varphi)) \neq g(\bar{v}(\psi))$

But then, taking $w := v \circ g$, we get a contradiction.

QUESTION: $\quad \left|\mathrm{Free}_{BA}(\mathcal{L})\right| = \,?$

$$\begin{tikzcd} BA \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & Set \arrow[l, bend right, "\mathrm{Free}_{BA}"'] \end{tikzcd}$$

$$\mathrm{Free}_{BA}(\mathcal{L}) = \mathrm{Free}_{BA}\left(\sum_{p\in\mathcal{L}}\{*\}\right) \overset{F \text{ is left adj}}{=} \sum_{p\in\mathcal{L}} \mathrm{Free}_{BA}(\{*\})$$

IT IS DUAL TO THE STONE SPACE

$$\prod_{p\in\mathcal{L}} \underbrace{\mathrm{Ult}\left(\mathrm{Free}_{BA}(\{*\})\right)}_{\substack{\cong \\ \hom_{BA}(\mathrm{Free}_{BA}(\{*\}),\,2)\ \cong\ \hom_{Set}(\{*\},\,2) = 2}} = \prod_{p\in\mathcal{L}} 2$$

~~IN CO~~ TO SUM UP, the Stone dual of $\mathrm{Free}_{BA}(\mathcal{L})$

is $\prod_{p \in \mathcal{L}} 2 = \underline{2^{\mathcal{L}}}$ with the product topology.

If $|\mathcal{L}| = n$

$$\Big|\underbrace{\mathrm{Free}_{BA}(\mathcal{L})}\Big| = \Big|\mathrm{Clop}\big(\underbrace{2^{\mathcal{L}}}_{\text{infinite}}\big)\Big| = \big|\mathcal{P}(2^{\mathcal{L}})\big| = 2^{2^{n}}.$$

$\downarrow$ PROPOSITIONAL INTUITIONISTIC LOGIC

FIRST-ORDER CLASSICAL LOGIC

NATURAL DUALITIES

$$
\begin{tikzcd}
\mathrm{Stone}^{op} \arrow[r, bend right, "\mathrm{Clop}(\ ) \simeq \hom(-,2)"'] & \mathrm{BA} \arrow[l, bend right, "\hom(-,2)"']
\end{tikzcd}
$$

$S \in \mathrm{Set}$

$$
\begin{tikzcd}
\mathrm{Set}^{op} \arrow[r, bend left, "\hom(-,S)"] & \mathrm{Set} \arrow[l, bend left, "\hom(-,S)"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{KHaus} \arrow[r, bend left, "\text{``}\hom(-,\mathbb{C})\text{''}"] & C^*\text{-alg} \arrow[l, bend left, "\text{``}\hom(-,\mathbb{C})\text{''}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Ab\,Group} \arrow[r, bend left, "\hom(-,\bigcirc)"] & \mathrm{Comp.\,Ab.\,Group} \arrow[l, bend left, "\hom(-,\bigcirc)"]
\end{tikzcd}
$$