2026.03.18

First order Boolean doctrines ~~(over FinSet^{op})~~

$$F : \mathrm{FinSet} \to \mathrm{BA}$$

$$\updownarrow$$

Polyadic spaces

certain $F : \mathrm{FinSet}^{op} \to \mathrm{Stone}$

Gödel's completeness theorem asserts that polyadic spaces are in fact spaces of models.

$\exists x \top$

$\exists x \top \not\equiv 1$ in context $\emptyset$.

$\forall y \exists x \top$

PROOF. Fix y, and let us prove $\underline{\exists x \top}$.

Take $x := y$.

$\exists x \top \equiv 1$ in context $\{y\}$

SEMANTIC EQUIVALENCE modulo a theory T in context X

$$\varphi \equiv^T_X \psi \iff \left(\begin{array}{c} \text{for every model } M \text{ of } T \text{ every function } \nu : X \to M \\ M, \nu \models \varphi \iff M, \nu \models \psi \end{array} \right)$$

It depends on the context

$\mathcal{L} = \emptyset$, $T = \emptyset$

$$\exists x \top \not\equiv^T_\emptyset \top \quad \text{but} \quad \exists x \top \equiv^T_{\{y\}} \top$$

FOBD : $P : \mathrm{FinSet} \to \mathrm{BA}$ functor s.t. $(y \notin X)$

(1) adding a dummy variable y to context X has a left adjoint $((\exists y)_X)$ and a right one $((\forall y)_X)$

$$\begin{tikzcd} \psi \in P(X) \arrow[rr, "P(X \hookrightarrow X \cup \{y\})"] & & P(X \cup \{y\}) \ni \varphi \arrow[ll, bend right, "(\exists y)_X"'] \arrow[ll, bend left, "(\forall y)_X"] \end{tikzcd}$$

$$(\exists y)_X \varphi \le \psi \iff \varphi \le P(X \cup \{y\})(\psi)$$

$$P(X \hookrightarrow X \cup \{y\})(\psi) \le \varphi \iff \psi \le (\forall y)_X \varphi$$

(2) Beck-Chevalley.
quantifiers commute with substitutions:

$$\begin{tikzcd} X \not\ni y \arrow[d, "f"] & P(X\cup\{y\}) \arrow[r, "(\exists y)_X"] \arrow[d, "P\left(\substack{x \in X \mapsto f(x) \\ y \mapsto y'}\right)"'] \arrow[dr, phantom, "="] & P(X) \arrow[d, "P(f)"] \\ X' \not\ni y' & P(X'\cup\{y'\}) \arrow[r, "(\exists y')_{X'}"'] & P(X') \end{tikzcd}$$

same for $\forall$.

Let P be a FOBD:

Let us prove $T_{P(\emptyset)} \overset{?}{\leq} (\forall y)_{\emptyset} (\exists x)_{\{y\}} T_{P(\{x,y\})}$ in $P(\emptyset)$ $\iff$ $P(\emptyset \hookrightarrow \{y\})(T_{P(\emptyset)}) \leq (\exists x)_{\{y\}} T_{P(\{x,y\})}$ in $P(\{y\})$

$L \dashv R \qquad 1 \leq RL$

$a \leq RLa$

$(\exists x)_z \dashv P(z \hookrightarrow z\cup\{x\}) \rightsquigarrow \varphi \leq P(z \hookrightarrow z\cup\{x\})((\exists x)_z \varphi)$

$\Updownarrow$

$(\exists x)_z \varphi \leq (\exists x)_z \varphi$

$$
\begin{tikzcd}
\emptyset \arrow[d] & y \\
\{y\}_{\not\ni x}
\end{tikzcd}
$$

$$
\begin{tikzcd}
T_{P(\{y\})} \arrow[rr, mapsto, bend left] \arrow[dd, mapsto, bend right] & P(\{y\}) \arrow[r, "(\exists y)_{\emptyset}"] \arrow[d, "P\left(\substack{\{y\}\to\{y,x\} \\ y\mapsto x}\right)"'] \arrow[dr, phantom, "="] & P(\emptyset) \arrow[d, "P(\emptyset\to\{y\})"] & (\exists y)_{\emptyset} T_{P(\{y\})} \arrow[d, mapsto] \\
& P(\{x,y\}) \arrow[r, "(\exists x)_{\{y\}}"'] & P(\{y\}) & P(\emptyset\to\{y\})(\exists y)_{\emptyset} T_{P(\{y\})} \\
T_{P(\{x,y\})} \arrow[rr, mapsto] & & (\exists x)_{\{y\}} T_{P(\{x,y\})} = P(\emptyset\to\{y\})(\exists y)_{\emptyset} T_{P(\{y\})}
\end{tikzcd}
$$

$$P(\emptyset \hookrightarrow \{y\})(T_{P(\emptyset)}) \overset{?}{\leq} (\exists x)_{\{y\}} T_{P(\{x,y\})}$$

in $P(\{y\})$

$$\Updownarrow$$

$$T_{P(\{y\})} \leq P(\emptyset \hookrightarrow \{y\})\left((\exists y)_{\emptyset} T_{P(\{y\})}\right) \checkmark$$

$$\Updownarrow$$

$$(\exists y)_{\emptyset} T_{P(\{y\})} \leq (\exists y)_{\emptyset} T_{P(\{y\})} \checkmark$$

If $\underset{\text{FinSet}}{X \hookrightarrow Y}$ is inj $\quad P(X \hookrightarrow Y)$ may fail to be inj.

$$P(\phi \hookrightarrow \{x\})(\exists x T) = P(\phi \hookrightarrow \{x\})(T)$$

even if $\exists x T \neq T$.

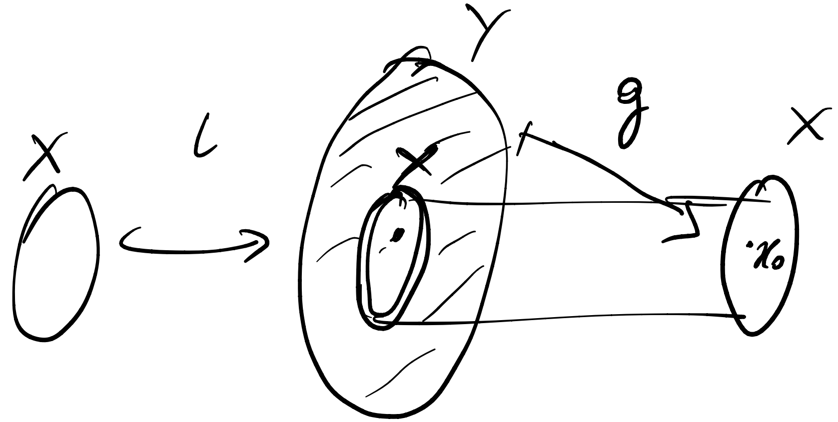
This can occur only if $X = \phi$.

i.e. $\begin{array}{l} X \neq \phi \\ X \underset{\text{inj}}{\hookrightarrow} Y \end{array} \Rightarrow P(X \hookrightarrow Y)$ is inj.

Proof. $X \hookrightarrow Y$

$\;\;\;\;\;\;\;\;\;\;\;\, \neq$

$\;\;\;\;\;\;\;\;\;\;\;\, \phi$



$$g \circ i = id_X$$

$$\Downarrow$$

$$P(g) \circ P(i) = P(g \circ i) = P(id_X) = id_{P(X)}$$

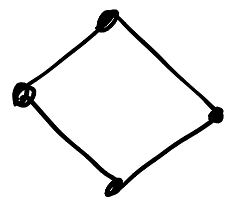
inj   surj.

$$\begin{tikzcd} P(X) \arrow[r, "P(i)"] \arrow[rr, bend right, equal] & P(Y) \arrow[r, "P(g)"] & P(X) \end{tikzcd}$$

EX. $\mathcal{L} = \emptyset$. $T = \emptyset$.

$LT_T : \mathrm{FinSet} \longrightarrow \mathrm{BA}$

$\emptyset \longmapsto \{1, 0, \exists x 1, \neg \exists x 1\}$



$X \longmapsto \{1, 0\}$

$X \neq \emptyset$

Given

$$\begin{array}{c} X \\ f\big\downarrow \\ Y \end{array}$$

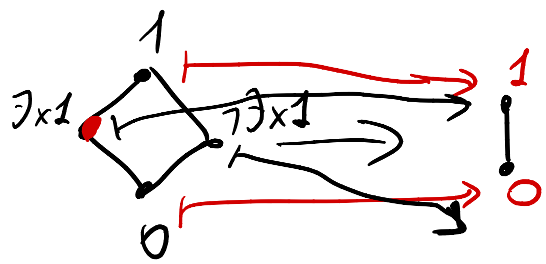
$$LT_\tau(f) : LT_\tau(X) \longrightarrow LT_\tau(Y)$$

if $X \neq \emptyset$ $\quad$ $\mathbb{I} \xrightarrow{id} \mathbb{I}$

if $X = \emptyset$, $Y = \emptyset$
then $f = id_\emptyset$ and so $LT_\tau(f) = id_\diamond$

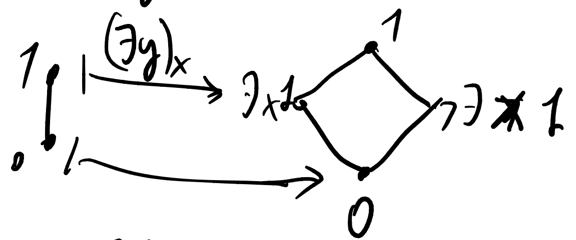
if $X = \emptyset$, $Y \neq \emptyset$

$LT_\tau(f)$:



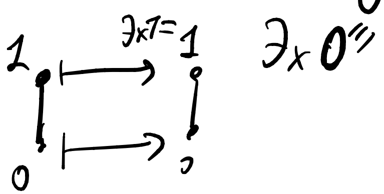
What is $(\exists y)_X : LT_\tau(X \cup \{y\}) \longrightarrow LT_\tau(X)$

if $X = \emptyset$ then



in the empty context
$\exists x 1 \not\equiv 1$

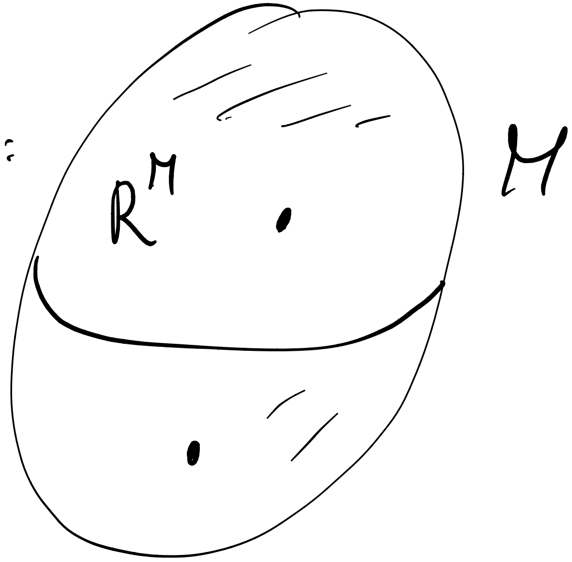
if $X \neq \emptyset$ then



EXAMPLE: $\mathcal{L} = \{R\}$, $T = \emptyset$

$ar(R) = 1$

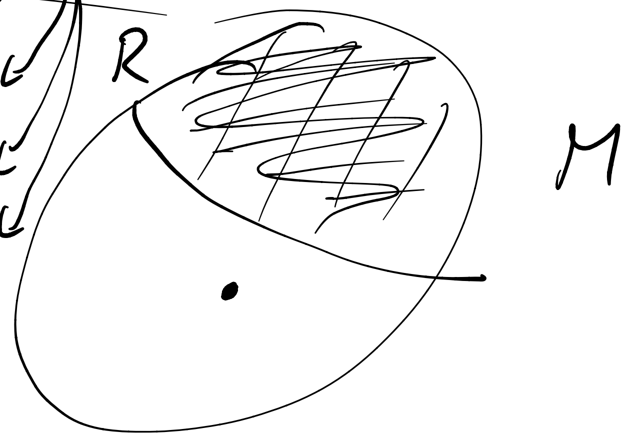
$LT_T(\emptyset) =$ is a Bool. alg. with four atoms:



- $(\neg \exists x\, R(x) \wedge \neg \exists x\, \neg R(x))$
- $(\neg \exists x\, R(x) \wedge \exists x\, \neg R(x))$
- $(\exists x\, R(x)) \wedge (\neg \exists x\, \neg R(x))$
- $(\exists x\, R(x)) \wedge (\exists x\, \neg R(x))$

- $LT_T(\{x\})$
  - $\to R(x)$
    - $\to R(x) \wedge \exists y\, \neg R(y)$
    - $\to R(x) \wedge \neg \exists y\, \neg R(y)$
  - $\to \neg R(x)$
    - $\to \neg R(x) \wedge \exists y\, R(y)$
    - $\to \neg R(x) \wedge \neg \exists y\, R(y)$

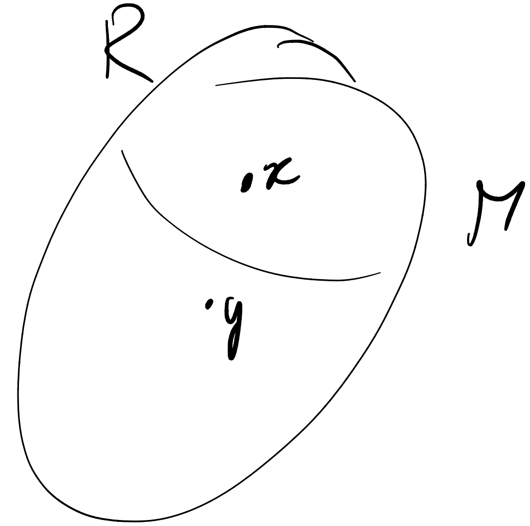
atoms



• $LT_T(\{x,y\}) \longrightarrow$ $R(x) \wedge \neg R(y)$

$R(y) \wedge \neg R(x)$

$\vdots$



CLASS. PROP. LOGIC

Model of a theory T in a lang $\mathcal{L}$

$v: \mathcal{L} \to 2$ satisfying all axioms in T

$\Updownarrow$ algebraically

ultrafilter of a Bool. alg.

CLASS. FO LOGIC

Given a language $\mathcal{L}$ and a theory T, we saw what is a model T.

i.e. $\mathcal{L}$-structure M s.t. $\forall \varphi \in T$

$M \models \varphi$.

$R^M \subseteq M^{ar(R)}$

$\Updownarrow$

Let us see what is a model of a FOBD.

DEF Let $P : \mathrm{FinSet} \to \mathrm{BA}$ be a FOBD.

A model of P consists of

- a set M
- For every set X, and every $\varphi \in P(X)$, we have a subset

$$\varphi^M$$

of $M^X = \{\text{functions } X \to M\}$

such that

$\alpha \in P(X)$

function symbols of arity $|X|$

(1) For every X,

- For all $\varphi, \psi \in P(X)$, $(\varphi \vee \psi)^M = \varphi^M \cup \psi^M$, and $(\varphi \wedge \psi)^M = \varphi^M \cap \psi^M$
- For all $\varphi \in P(X)$ $(\neg\varphi)^M = M^X \setminus \varphi^M$
- $0^M = \emptyset$, $1^M = M^X$

(2) For every morphism $f : X \to Y$ in FinSet,

$$\Big(\underbrace{\underbrace{P(f)(\varphi)}_{}}_{\subseteq M^Y,\ \in P(Y)}\Big)^M = \{ g \in M^Y \mid g \circ f \in \varphi^M \}$$

$$X \xrightarrow{f} Y \xrightarrow{g} M$$

(3) • For every $X \in \mathrm{FinSet}$, every $y \notin X$, every $\varphi \in P(X \cup \{y\})$

$$\Big(\underbrace{\underbrace{\exists x\, \varphi}_{\in P(X)}}_{\subseteq M^X}\Big)^M = \left\{ f \in M^X \;\middle|\; \exists a \in M : \left(\begin{matrix} x \overset{\in X}{\mapsto} f(x) \\ y \mapsto a \end{matrix} \; X \cup \{y\} \to M \right) \in \varphi^M \right\}$$

$f : X \to M$

$$\underbrace{\varphi^M}_{\in P(X \cup \{y\})} \subseteq M^{X \cup \{y\}}$$

equivalently

$$(\forall x\, \varphi)^M = \left\{ f \in M^X \;\middle|\; \forall a \in M : \left(\begin{matrix} x \overset{\in X}{\mapsto} f(x) \\ y \mapsto a \end{matrix} \; X \cup \{y\} \to M \right) \in \varphi^M \right\}$$

(End of def. of model) $\triangleright$

DEF **Morphisms of FOBD**

A morphism of FOBD from $P_1 : \mathrm{FinSet} \to BA$ to $P_2 : \mathrm{FinSet} \to BA$ is a natural transformation

$$m : P_1 \longrightarrow P_2$$

$$\begin{tikzcd}
X \arrow[d, "f"'] & P_1(X) \arrow[r, "m_X"] \arrow[d, "P_1(f)"'] \arrow[dr, phantom, "//"] & P_2(X) \arrow[d, "P_2(f)"] & \text{Bool. hom.} \\
Y & P_1(Y) \arrow[r, "m_Y"'] & P_2(Y) &
\end{tikzcd}$$

s.t. for all $X \in \mathrm{FinSet}$, $y \notin X$

$$\begin{tikzcd}
P_1(X \cup \{y\}) \arrow[r, "(\exists y)_X"] \arrow[d, "m_{X \cup \{y\}}"'] & P_1(X) \arrow[d, "m_X"] \\
P_2(X \cup \{y\}) \arrow[r, "(\exists y)_X"'] & P_2(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
P_1(X \cup \{y\}) \arrow[r, "(\forall y)_X"] \arrow[d, "m_{X \cup \{y\}}"'] & P_1(X) \arrow[d, "m_X"] \\
P_2(X \cup \{y\}) \arrow[r, "(\forall y)_X"'] & P_2(X)
\end{tikzcd}$$

Let $P : \mathrm{FinSet} \to \mathrm{BA}$ be a FOBD.

A model of P consists of

- a set M
- a morphism of FOBD from $P : \mathrm{FinSet} \to \mathrm{BA}$ to

$$\mathrm{FinSet} \xrightarrow[\hom(-,M)]{} \mathrm{Set}^{op} \xrightarrow{\mathcal{P}} \mathrm{BA}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & M^X \arrow[r, mapsto] & \mathcal{P}(M^X) \arrow[d] & A \subseteq M^X \arrow[d, mapsto] \\
Y \arrow[r, mapsto] & M^Y \arrow[u, "g \circ f \,\hat{\mathrel{|}}\, g"'] \arrow[r, mapsto] & \mathcal{P}(M^Y) & \{ g \in M^Y \mid g \circ f \in A \}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Set}^{op} \arrow[r] & \mathrm{BA} \\
X \arrow[r, mapsto] & \mathcal{P}(X) \arrow[d, "f^{-1}[-]"] \\
Y \arrow[u, "f"] \arrow[r, mapsto] & \mathcal{P}(Y)
\end{tikzcd}$$

EX: it is a FOBD.

$$\begin{tikzcd}
\mathrm{FinSet} \arrow[r, shift left, "P"] \arrow[r, shift right, "\mathcal{P} \circ \hom(-,M)"'] & \mathrm{BA}
\end{tikzcd}$$

$$\varphi \longmapsto \varphi^M \subseteq M^X$$

$$\overset{\varphi}{\cap}$$

$$m_X : P(X) \longrightarrow \mathcal{P}(M^X)$$

Gödel's completeness theorem: classical formulation: if a theory does not prove $\bot$, then it has a model.

THM GÖDEL'S COMPLETENESS THEOREM (FOR FIRST-ORDER BOOLEAN DOCTRINES) (Blackbox)
Let $P: \mathrm{FinSet} \to \mathrm{BA}$ be a FOBD. If, in the Bool. alg. $P(\emptyset)$, $0 \neq 1$, then P has a model.

This is a first-order version of the Boolean Prime Ideal Theorem.

COROLLARY
Let $P: \mathrm{FinSet} \to \mathrm{BA}$ be a FOBD. Let F be a filter of $P(\emptyset)$, and I be an ideal of $P(\emptyset)$, and suppose $F \cap I = \emptyset$. Then, there is a model M of P s.t.

- for every $\varphi \in F$, $M \models \varphi$
- for every $\varphi \in I$, $M \not\models \varphi$

NOTATION Given a FOBD $P: \mathrm{FinSet} \to \mathrm{BA}$, a model M of P, and $\varphi \in P(\emptyset)$, we write
$$M \models \varphi$$
if φ^M is the singleton $M^\emptyset$.
$$\cong M^\emptyset$$

Before proving the corollary, we introduce the filter-quotient construction.

PROP Let $P : \mathrm{FinSet} \to \mathrm{BA}$ be a FOBD. Let G be a filter of $P(\emptyset)$.

Then, the following defines a FOBD

$$G \subseteq P(\emptyset) \qquad \emptyset \to X$$

$$\begin{array}{ccc} G \subseteq P(\emptyset) & \to & P(X) \\ \vartheta & \longmapsto & \varphi, \psi \end{array}$$

$P/_G : \mathrm{FinSet} \to \mathrm{BA}$

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & P(X)/\sim_X \arrow[d] & {[\varphi]} \arrow[d, mapsto] \\ Y \arrow[r, mapsto] & P(Y)/\sim_Y & {[P(f)(\varphi)]} \end{tikzcd}$$

, where $\varphi \sim_X \psi \iff \exists \vartheta \in G :$ $\varphi, \psi \in P(X)$

$$P(\emptyset \hookrightarrow X)(\vartheta) \leq \varphi \leftrightarrow \psi.$$

$$\rightsquigarrow [\varphi] \wedge [\psi] = [\varphi \wedge \psi]$$

$$(\exists x)_X [\varphi] = [(\exists y)_X \varphi]$$

$\forall$ $\qquad\qquad$ $\forall$.