Def Let $P: \mathrm{FinSet} \to \mathrm{BA}$ be a FOBD.

Let M, N be two models of P.

We write $M \equiv N$ if, for every $\varphi \in P(\emptyset)$,

elem. equiv.

$$M \models \varphi \iff N \models \varphi.$$

The equiv. class of a model is called a Type.

$\mathrm{Mod}(P)$ = class of models of P

$\mathrm{Typ}(P)$ = set of Types of P = $\mathrm{Mod}(P)/\equiv$

$$\boxed{\begin{array}{l}\text{THM Let } P \text{ be a FOBD. The following defines a bijection} \\ \qquad \mathrm{Typ}(P) \longrightarrow \mathrm{Ult}(P(\emptyset)) \\ \qquad [M] \longmapsto \{\varphi \in P(\emptyset) \mid M \models \varphi\}\end{array}}$$

It does not depend on the choice of M for its equiv. class.

PROOF 1) It is easy to see that $\{\varphi \in P(\emptyset) \mid M \models \varphi\}$ is an ultrafilter.

For example, to see that it is closed under $\wedge$.

$$M \models \varphi, \; M \models \psi \overset{?}{\Longrightarrow} M \models \varphi \wedge \psi \quad \checkmark$$

$\varphi^M = M^\emptyset \quad \psi^M = M^\emptyset \qquad (\varphi \wedge \psi)^M$ is the singleton $M^\emptyset$

$$(\varphi \wedge \psi)^M = \varphi^M \cap \psi^M = M^\emptyset \cap M^\emptyset = M^\emptyset \qquad \text{EXERCISE.}$$

2) INJ: just follows from the def. of elem. equivalence.

3) SURJ: Let U be an ultrafilter of $P(\emptyset)$.

We shall produce a model M of P such that

$$U = \{\varphi \in P(\emptyset) \mid M \models \varphi\}.$$

I instantiate the corollary with $F := U$, $I := P(\emptyset) \setminus F$.

This gives me a model M s.t.

$$\forall \varphi \in U \quad M \models \varphi$$
$$\forall \varphi \in P(\emptyset) \setminus U \quad M \not\models \varphi.$$

BTW: Gödel's compl. thm uses the ax. of choice

<u>Def</u> Let $P : \mathrm{FinSet} \to \mathrm{BA}$ be FOBD.
We equip $\mathrm{Typ}(P)$ with the topology generated by the sets

$$[\![\varphi]\!] := \{ [M] \in \mathrm{Typ}(P) \mid M \models \varphi \}, \quad \varphi \in P(\emptyset)$$

<u>LEM</u> let $\Gamma : \mathrm{Typ}(P) \to \mathrm{Ult}(P(\emptyset))$ be the bijection above.
For every $\varphi \in P(\emptyset)$, $\Gamma^{-1}(\eta_{P(\emptyset)}(\varphi)) = [\![\varphi]\!]$.

<u>PROOF</u> $\Gamma^{-1}(\eta_{P(\emptyset)}(\varphi)) = \{ [M] \in \mathrm{Typ}(P) \mid \Gamma([M]) \in \eta_{P(\emptyset)}(\varphi) \} =$

$$= \{ [M] \in \mathrm{Typ}(P) \mid \varphi \in \Gamma([M]) \}$$

$$\parallel$$

$$= \{ [M] \mid M \models \varphi \} = [\![\varphi]\!] .$$

<u>THM</u> Let $P : \widehat{\mathrm{FinSet}} \to \mathrm{BA}$ be a FOBD

The bij. $\mathrm{Typ}(P) \to \mathrm{Ult}(P(\emptyset))$ is a homeomorphism.

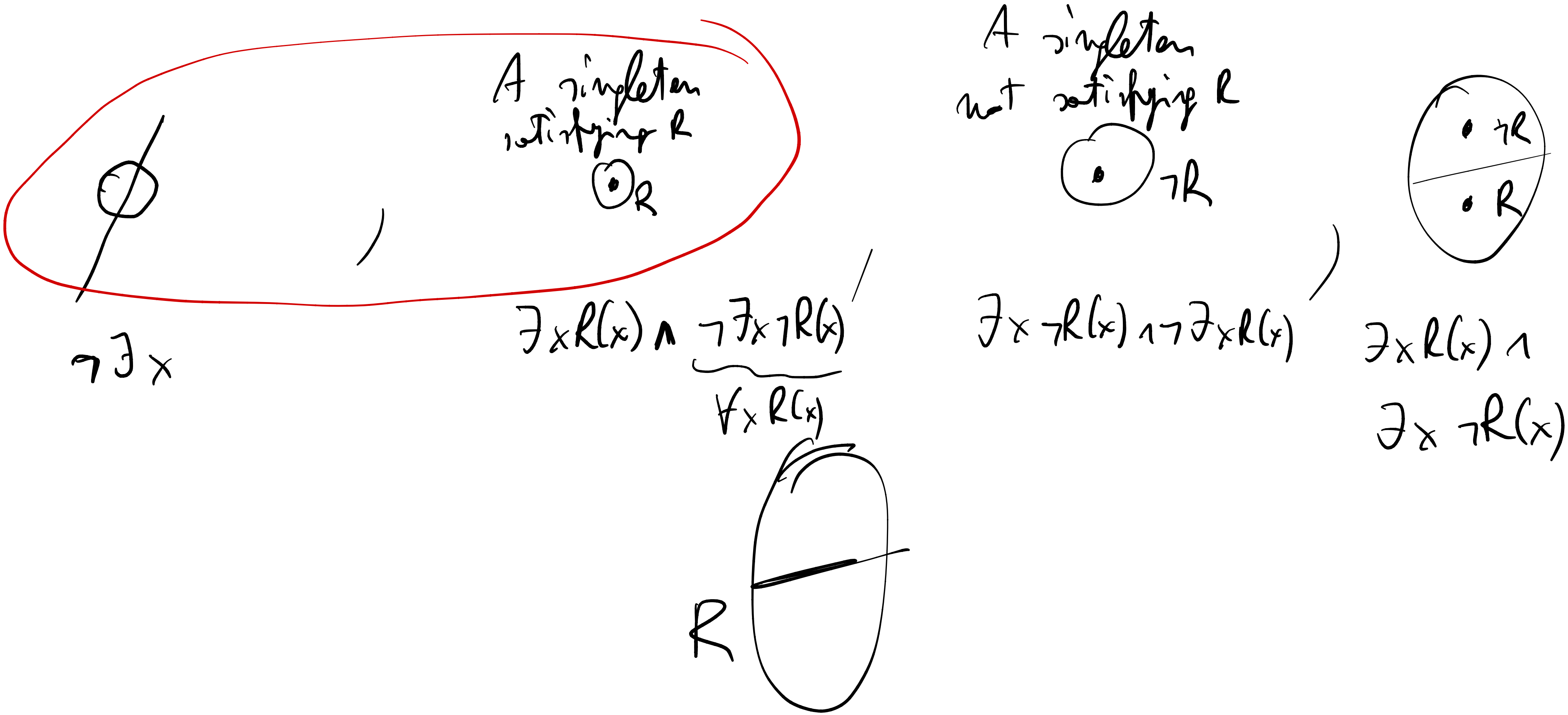
Therefore, $\mathrm{Typ}(P)$ is a Stone space, whose clopens are the sets

$$[\![\varphi]\!] := \{ [M] \mid M \models \varphi \} \qquad \varphi \in P(\emptyset) .$$

<u>Proof</u> By the lemma above.

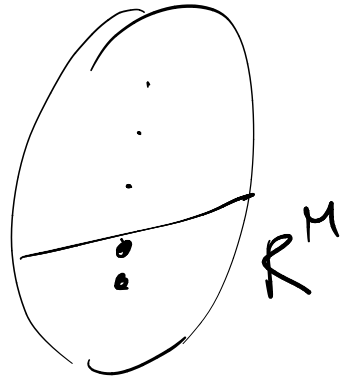
In conclusion, the space of types is the Stone dual of the Bool. alg. of sentences.

E.g.: Consider the language $\mathcal{L}_0=\{R\}$, and the theory $T=\emptyset$.
Up to elementary equiv., these are the models:



$LT_\emptyset(\emptyset)$ in the Bool. alg.

$$P(4)$$



$$(\neg \exists x) \vee (\exists x R(x) \wedge \neg \exists \neg R(x)) \equiv \forall x R(x)$$

What is the Stone dual of $P(X)$.

Def Let $\mathcal{L}$ be a language and T a Theory, and X a finite set of variables.

An X-pointed model of T consists of

- A model M of T, and
- a function $\nu : X \to M$.

Two X-pointed models $(M,\nu), (M',\nu')$ are elementarily equivalent

denoted $(M,\nu) \equiv (M',\nu')$

if for every formula φ s.t. $FV(\varphi) \subseteq X$,

$$M,\nu \models \varphi \iff M',\nu' \models \varphi$$

Give me a model M

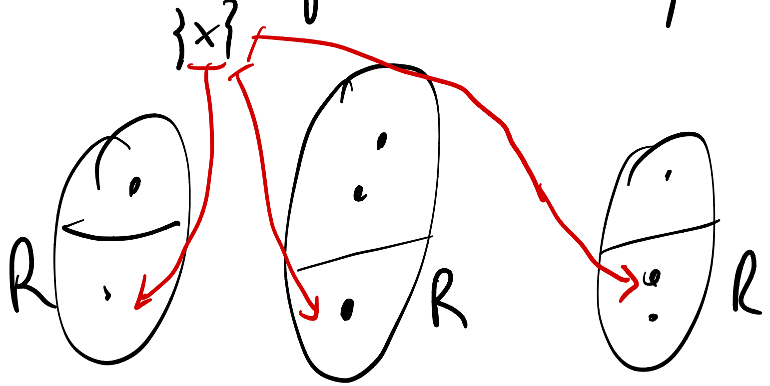
$X = \{x\}$

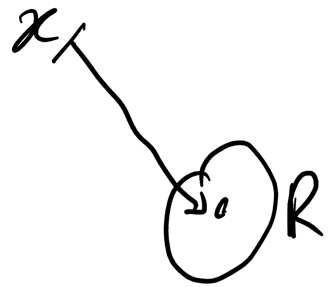
ν, ν' s.t. $(M,\nu) \models \varphi$

$(M,\nu') \not\models \varphi$

Q: is it the same as $M \equiv M'$?

Def An *X-pointed type* (or simply, an X-type) is the equiv. class of an X-pointed model.





$\exists y \neg R(y)$

$\{x\}$

Def Let $P : \mathrm{FinSet} \to BA$ be a FOBD, let $X \in \mathrm{FinSet}$.

An X-pointed model of P consists of

- A model M of P
- A function $\nu : X \to M$.

Def For (M,ν), (M',ν') X-pointed models of P, we say that (M,ν), (M',ν') are *elementarily equivalent*,

denoted $(M,\nu) \equiv (M',\nu')$

if, for every $\varphi \in P(X)$,

$$M,\nu \vDash \varphi \iff M',\nu' \vDash \varphi.$$

Def $M,\nu \vDash \varphi$ means $\underset{X \to M}{\nu} \in \underbrace{\varphi^M}_{\subseteq M^X}$.

An X-pointed Type of P (X-Type) is the equivalence class of X-pointed models.

THM Let $P: \mathrm{FinSet} \to \mathrm{BA}$ be a FOBD. Let $X \in \mathrm{FinSet}$.
Let F be a filter of $P(X)$, and I ideal of $P(X)$ s.t. $F \cap I = \emptyset$.
There is an X-pointed model $(M, \underset{X \to M}{\nu})$ of P such that:

- for all $\varphi \in F$, $M, \nu \models \varphi$
- " " $\varphi \in I$, $M, \nu \not\models \varphi$.

BLACK-BOX.

Given a FOBD $P: \mathrm{FinSet} \to \mathrm{BA}$ and $X \in \mathrm{FinSet}$
we denote by

$\mathrm{Mod}_X(P) :=$ class of X-pointed models of P.

$$\mathrm{Typ}_X(P) = \mathrm{Mod}_X(P) /_{\equiv}$$

Equip $\mathrm{Typ}_X(P)$ with the topology generated by

$$[\![\varphi]\!] = \{ [(M,\nu)] \in \mathrm{Typ}_X(P) \mid M,\nu \models \varphi \} \quad \text{for } \varphi \in P(X).$$

<u>THM</u> Let $P : \mathrm{FinSet} \to \mathrm{BA}$ be a FOBD, let $X \in \mathrm{FinSet}$. The function

$$\begin{aligned} \mathrm{Typ}_X(P) &\longrightarrow \mathrm{Ult}(P(X)) \\ [(M,\nu)] &\longmapsto \{ \varphi \in P(X) \mid M,\nu \models \varphi \} \end{aligned}$$

is a homeomorphism.

Thus, $\mathrm{Typ}_X(P)$ is a Stone space, and is the STONE dual of $P(X)$.

<u>PROOF</u> Exercise.

INJ. is immediate

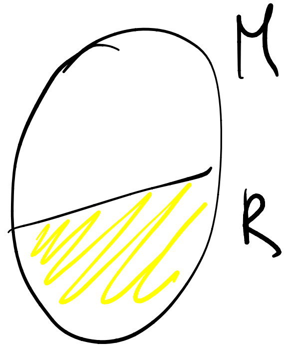
SURJ: Use GÖDEL'S COMPL. THM. (more sophisticated version, with $X \in \mathrm{FinSet}$)

homeom.

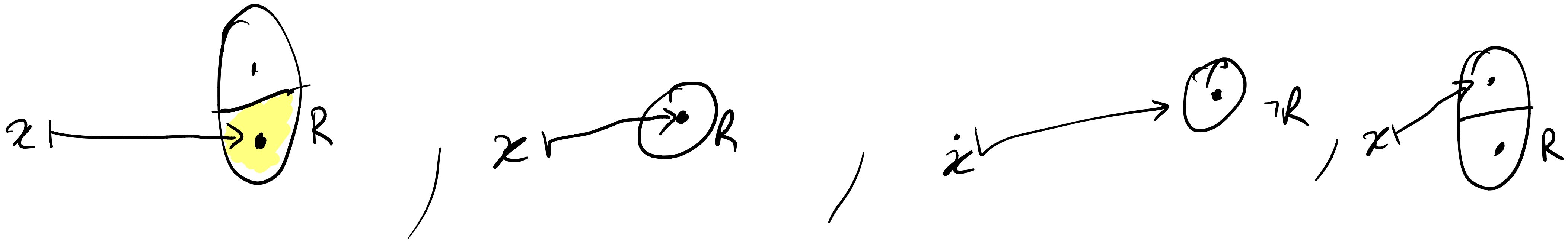
E.G. $\mathcal{L} = \{R\}$, $T = \emptyset$.

unary

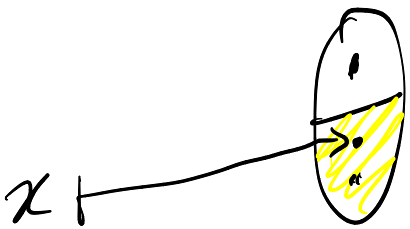
$\{x\}$



$\{x\}$-Types?



$\forall y R(y)$



$f : X \longrightarrow Y$
$\quad\;\, \cap \qquad\quad \cap$
$\quad \mathrm{FinSet}$

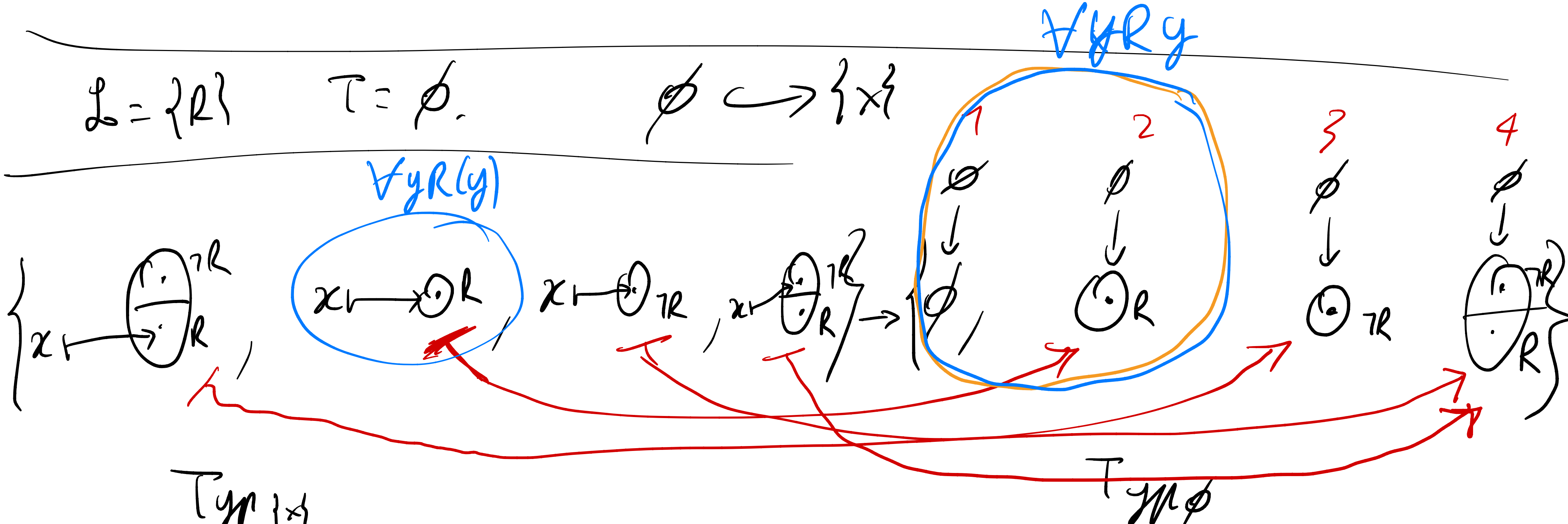
φ FOBD.

$$\mathrm{Typ}_Y(\varphi) \longrightarrow \mathrm{Typ}_X(\varphi)$$

$$[(M, \underbrace{\nu}_{Y \to M})] \longmapsto [(M, \underbrace{\nu \circ f}_{X \to M})]$$

$$X \xrightarrow{f} Y \xrightarrow{\nu} M$$

$\mathcal{L} = \{R\} \qquad T = \emptyset. \qquad \emptyset \hookrightarrow \{x\}$



$$\mathrm{Typ}_Y(P) \longrightarrow \mathrm{Typ}_X(P)$$

$$[(M,\nu)] \longmapsto [(M,\nu\circ f)]$$

$\nu : Y \to M$, $\quad \nu\circ f : X \to M$

$$f : X \longrightarrow Y$$

$$Pf : P(X) \longrightarrow P(Y)$$

$$\varphi \mapsto P(f)(\varphi)$$

Well-defined?

$$(M,\nu) \equiv (M',\nu') \overset{?}{\Longrightarrow} (M,\nu\circ f) \equiv (M',\nu'\circ f)$$

$\nu : Y \to M$, $\quad \nu' : Y \to M'$, $\quad \nu\circ f : X \to M$, $\quad \nu'\circ f : X \to M'$

$\Updownarrow$

$\forall \psi \in P(Y)$

$M,\nu \models \psi \iff M',\nu' \models \psi$

i.e.

$\nu \in \psi^M \iff \nu' \in \psi^{M'}$

$\Updownarrow$

$\forall \varphi \in P(X) \quad M, \nu\circ f \models \varphi \iff M', \nu'\circ f \models \varphi$

Fix φ, and let us prove $\Rightarrow$

$M, \nu\circ f \models \varphi$, i.e. $\nu\circ f \in \varphi^M$

$\Updownarrow$

$\nu \in P(f)(\varphi)^M = \{ w \mid w\circ f \in \varphi^M \}$

$\Downarrow$

$\nu' \in P(f)(\varphi)^{M'}$

$\vdots$

$M', \nu' \circ f \models \varphi.$

$$\mathrm{Typ}_f(P) : \mathrm{Typ}_Y(P) \longrightarrow \mathrm{Typ}_X(P)$$

$$[(M, \underbrace{\nu}_{Y \to M})] \longmapsto [(M, \underbrace{\nu \circ f}_{X \to M})]$$

CLAIM: this function is continuous.

ENOUGH: $\mathrm{Typ}_f(P)^{-1}(\llbracket \varphi \rrbracket) = \llbracket P(f)(\varphi) \rrbracket$, $\varphi \in P(X)$.

EXERCISE.

$$\begin{tikzcd}
X \arrow[d, "f"'] & \mathrm{Typ}_X(P) \arrow[r, "\sim"] \arrow[dr, phantom, "//"] & \mathrm{Ult}(P(X)) & P(X) \arrow[d, "P(f)"] \\
Y & \underline{\mathrm{Typ}_Y(P)} \arrow[u, "\mathrm{Typ}_f(P)"] \arrow[r, "\sim"] & \mathrm{Ult}(P(Y)) \arrow[u, "P(f)^{-1}[-]"'] & P(Y)
\end{tikzcd}$$

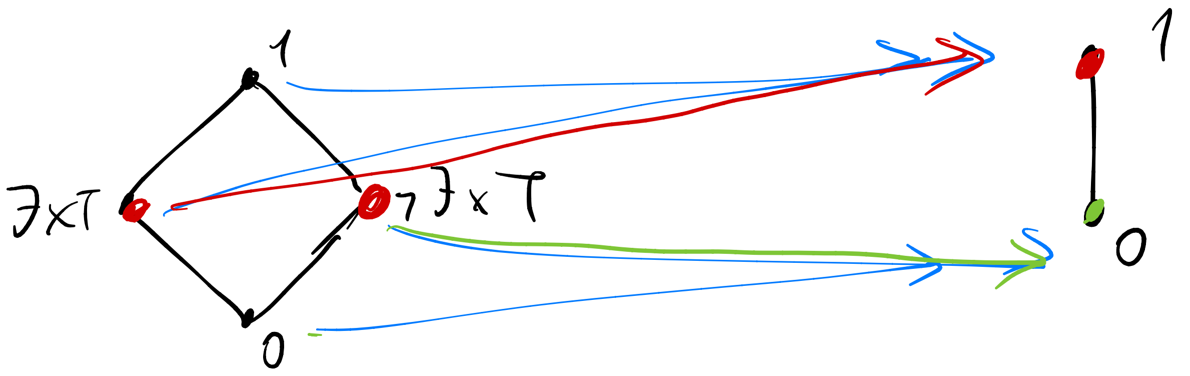
$$\begin{tikzcd}
{[M, \nu \circ f]} \arrow[r, mapsto] & \{\varphi \in P(X) \mid M, \nu \circ f \models \varphi\} \\
& \{\varphi \in P(X) \mid M, \nu \models P(f)(\varphi)\} \\
{[M, \nu : Y \to M]} \arrow[uu, mapsto] \arrow[r, mapsto] & \{\varphi \in P(Y) \mid M, \nu \models \varphi\} \arrow[u, mapsto]
\end{tikzcd}$$

CONCLUSION: Given a FOBD $P : \mathrm{FinSet} \to \mathrm{BA}$ we have a functor $\mathrm{Typ}_-(P) : \mathrm{FinSet}^{op} \to \mathrm{Stone}$ /

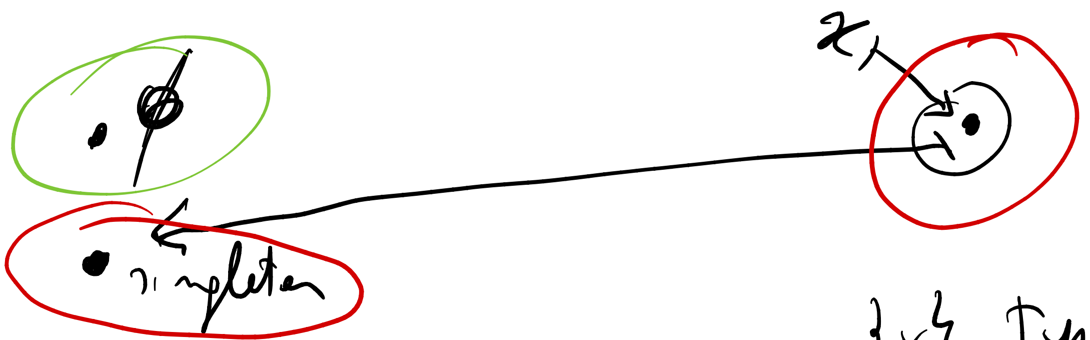
and it is naturally isomorphic to (the opposite functor of)

$$\mathrm{FinSet} \xrightarrow{P} BA \xrightarrow{\mathrm{Stone}} \mathrm{Stone}^{op}$$

EXAMPLE: $\mathcal{L} = \emptyset$, $T = \emptyset$



$LT(\emptyset)$ $LT(\{x\})$



$\emptyset$-Typ $\{x\}$-Type