$$\begin{tikzcd}
\text{FOBD} \arrow[r, bend left, "ult^*"] \arrow[r, "12", "Typ_-(P)"'] & \text{POLYADIC SPACE} \arrow[l, bend left, "Clop^*"]
\end{tikzcd}$$

$P : FinSet \to BA$

$E : FinSet^{op} \to Stone$.

$X \hookrightarrow X \cup \{y\}$

$E(X \cup \{y\}) \to E(X)$ is open

$$\begin{tikzcd}
X' \cup \{y'\} & X' \arrow[l, hook'] \\
X \cup \{y\} \arrow[u, "\tilde{\sigma}"] \arrow[ur, phantom, "\lrcorner"] & X \arrow[l, hook'] \arrow[u, "\sigma"']
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r, mapsto] \arrow[ddd, bend right] & b \arrow[ddd, bend left] \\
E(X' \cup \{y'\}) \arrow[r] \arrow[d, "E(\tilde{\sigma})"'] & E(X') \arrow[d, "E(\sigma)"] \\
E(X \cup \{y\}) \arrow[r] & E(X) \\
a \arrow[r, mapsto] & ''
\end{tikzcd}$$

quasi-pullback

EXAMPLES:

For M finite set

$$\begin{tikzcd}
\mathrm{FinSet}^{op} \arrow[r] & \mathrm{Stone} \\
X \arrow[r, mapsto] & \{f : X \to M\} \arrow[d, "- \circ \sigma"] \\
Y \arrow[u, "\sigma"] & {}
\end{tikzcd}$$

CLASSICAL PROPOSITIONAL LOGIC

$\vee, \wedge, \neg, 0, 1$

$\underbrace{\text{Prop. lang. } \mathcal{L}_0}_{\text{generators}}, \underbrace{\text{theory } T}_{\text{relations}}$

CLASSICAL FO LOGIC

$\vee, \wedge, \neg, 0, 1, \exists, \forall$

FO lang. $\mathcal{L}_0$, Theory T.

LET'S ALGEBRAIZE

identify equiv. formulas.

Bool. alg. | FOBD.

Group = A set equipped with a binary operation · a unary oper. $(-)^{-1}$ and a nullary op. 1 such that . . .

$$\mathbb{Z} = \langle * \mid \emptyset \rangle = \langle *, *' \mid *' = * \cdot * \rangle$$

$$\mathbb{Q} = \langle \{ \underset{\substack{\uparrow \\ \frac{1}{n!}}}{q_n} \mid n \in \mathbb{N} \setminus \{0\} \} \mid (q_{n+1})^{n+1} = q_n \rangle$$

$$= \langle \underset{\rightsquigarrow \frac{1}{n}}{i_n} \quad \mid \quad \rangle$$

PROPOSITIONAL

$\langle \underbrace{a, b}_{\mathcal{L}_0} \mid \emptyset \rangle$

$\langle \underbrace{c_1, c_2, c_3, c_4} \mid \begin{matrix} c_i \wedge c_j = 0 \text{ for } i \neq j \\ c_1 \vee \dots \vee c_4 = 1 \end{matrix} \rangle$

Free(2) $\rightsquigarrow$ $2^{2} = 4$ 2^{4} elements.

Language and theories are presentations, while Bool alg. are the actual mathem. object.

Bool. algs in syntax-free logic.

Giving a Bool. alg = giving a lang. and a Theory, taking all formulas up to equiv. (forgetting lang. + theory).

Giving a Stone space = give $\mathcal{L}_0$ + T,
taking the space of models.

Stone spaces is the mathematical structure of spaces of models.
Polyadic spaces " " Types.

CATEG. DUALITIES are a fine-tuned version of representation thms.

PROP. CASE

For every Bool. alg. B there is a set X and an embedding

$$B \hookrightarrow \mathcal{P}(X)$$

1, $\wedge$ intersections
0, $\vee$ unions
$\neg$ complement

FO CASE

For every FOBD P, there is a polyadic set

$$E : \mathrm{FinSet}^{op} \to \mathrm{Set}$$

and an embedding

$$P \hookrightarrow (\mathcal{P} \circ E^{op})$$

$$\mathrm{FinSet} \xrightarrow{E^{op}} \mathrm{Set}^{op} \xrightarrow{\mathcal{P}} \mathrm{BA}$$

1, $\wedge$ inter.
0, $\vee$ union
$\neg$ complement
$\exists$ direct image function

POLYADIC SET

$E : \mathrm{FinSet}^{o} \longrightarrow \mathrm{Set}$ s.t.

$$\begin{tikzcd}
\bullet \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & b \arrow[ddd, mapsto] \\
& E(X'\cup\{y'\}) \arrow[r] \arrow[d, "E(\tilde{\sigma})"'] & E(X') \arrow[d, "E(\sigma)"] & \\
& E(X\cup\{y\}) \arrow[r] & E(X) & \\
\overset{\varphi}{a} \arrow[rrr, mapsto] & & & ''
\end{tikzcd}$$

quasi-pullback

DUALITY: gives a representation of morphisms.

BA : complete + cocomplete

LIMITS are as in Set.

Stone : complete + cocomplete

LIMITS as in Set

FINITE COPROD. as in Set

FOBD : complete and cocomplete.

LIMITS: computed fiberwise.

TERMINAL : $\mathrm{FinSet} \to \mathrm{BA}$

$X \mapsto 1$

POLYADIC SPACES: complete and cocomplete

LIMITS "fiberwise"?

E.g. in the Terminal object

$1 : \mathrm{FinSet}^{op} \to \mathrm{Stone}$?

$X \mapsto 1$

Is it true that for every other $E : \mathrm{FinSet}^{op} \to \mathrm{Stone}$ there is a unique morphism to 1? NO

TERMINAL OBJECT : $\mathrm{FinSet}^{op} \to \mathrm{Stone}$

$$X \mapsto \begin{cases} 2 = \{a,b\} & X = \emptyset \\ 1 & X \neq \emptyset \end{cases}$$

$$\begin{tikzcd} 1 \arrow[r] & 2 \\ * \arrow[r, mapsto] & a \end{tikzcd}$$

- van Gool, Marquès, On duality and model theory for polyadic spaces.
- Reggio, Polyadic sets and homomorphisms counting.

FILTER-QUOTIENT CONSTRUCTION (DUALLY)

$P: \mathrm{FinSet}^{op} \longrightarrow BA$, Take a filter F of $P(\emptyset)$ then you get a certain FO-Bool. doctrine $P/_F$

$$P/_F(x) := P(x)/_{\sim_x} \qquad \varphi \sim_x \psi \iff \exists \vartheta \in F :$$

$$P(\emptyset \hookrightarrow x)(\vartheta) \leq \varphi \leftrightarrow \psi.$$

Let $E : \mathrm{FinSet}^{op} \to BA$ polyadic space.

Let $C \subseteq E(\emptyset)$ closed

$$\begin{tikzcd} \emptyset \arrow[r, hook] & X \\ E(X) \arrow[r] & E(\emptyset) \\ & C \arrow[u, phantom, sloped, "\subseteq"] \end{tikzcd}$$

Then

$\tilde{E} : \mathrm{FinSet}^{op} \to \mathrm{Stone}$

$$\begin{tikzcd} X \arrow[r, mapsto] & E(\emptyset \hookrightarrow X)^{-1}[C] \arrow[d, "\tilde{E}(\sigma)"] \\ Y \arrow[u, "\sigma"] \arrow[r, mapsto] & E(\emptyset \hookrightarrow Y)^{-1}[C] \end{tikzcd}$$

LEMMA:

Let X be a Stone space, there is a bijective correspondence

Filters of $\mathrm{Clop}(X) \longleftrightarrow$ Closed subsets of X

$\{A \in \mathrm{Clop}(X) \mid F \subseteq A\} \longleftarrow\!\shortmid F$

$\mathcal{F} \longmapsto \bigcap \mathcal{F}$

EXERCISE

$\tilde{\sigma}(\sigma)$ is the restriction of $E(\sigma)$

$$\begin{tikzcd} \tilde{E}(X) \arrow[r, hook] \arrow[d, "\sim"'] & E(X) \arrow[r] \arrow[d, "E(\sigma)"] & E(\emptyset) \subseteq C \\ \tilde{E}(X) \arrow[r, hook] & E(Y) \arrow[ur] & \end{tikzcd}$$

EXERCISE:

This is a polyadic space.

CONSEQUENCE

Every polyadic space with $E(\emptyset)$ finite is a disjoint union of polyadic spaces with $E(\emptyset)$ singleton.

—

Finite coproducts of polyadic spaces is disjoint union

Proof Directly, or through FOBD (products are fiberwise products).

Dually, every FOBD with $P(\emptyset)$ finite is a fiberwise direct product of FOBD with $P(\emptyset)$ a 2-elem. Bool. algebra.

$$X \longmapsto M^X$$

$FinSet^{op} \longrightarrow Stone$

$X \longmapsto \{\text{equiv. relation on } X\}$ with the discrete topology.

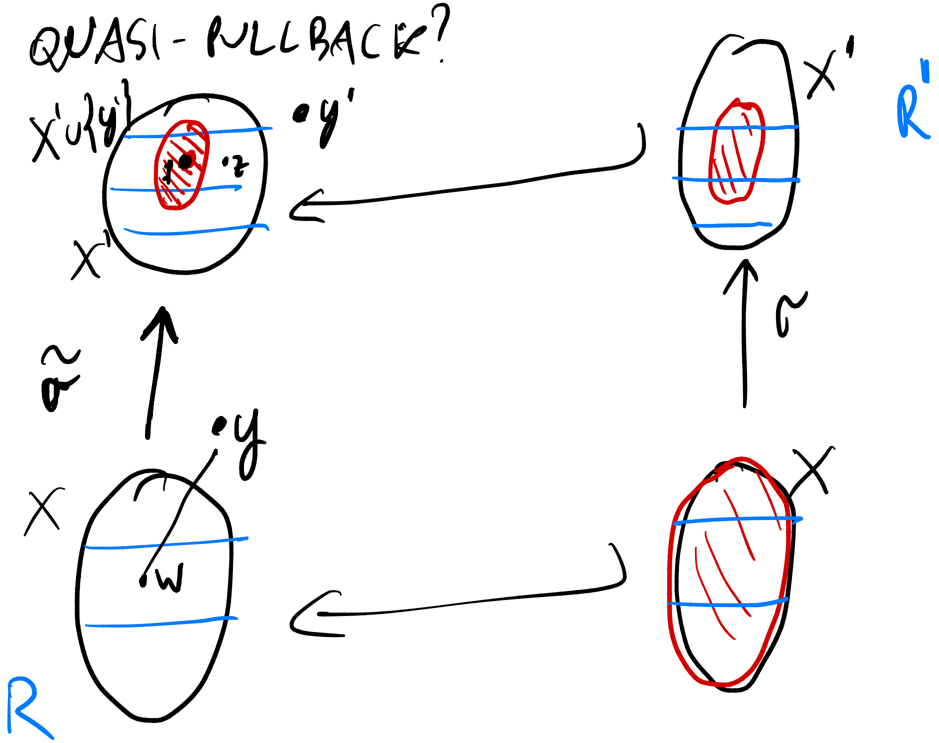
$$\begin{tikzcd} Y & \{\text{equiv. relations on } Y\} \ni R \arrow[d] & R \arrow[d, mapsto] \\ X \arrow[u, "\sigma"] & \{\text{equiv. relation on } X\} & R' \end{tikzcd}$$

$x_1 R' x_2 \iff \sigma(x_1) R \sigma(x_2)$

It's a polyadic space?

① OPENNESS is satisfied because the spaces are discrete

② QUASI-PULLBACK?



R equiv. relat. on $X \cup \{y\}$

R' equiv. relat. on X'

$$R|_X = (\sigma \times \sigma)^{-1}[R']$$

i.e. $\forall x_1, x_2 \in X$, $x_1 R x_2$

$$\Updownarrow$$

$\sigma(x_1) R' \sigma(x_2)$

I should create $\tilde{R}$ eq. rel. on $X' \cup \{y\}$ s.t.

- the restriction of $\tilde{R}$ to X' should be R'
- the eq. relation induced by $\tilde{R}$ on X under $\tilde{\sigma}$ should be R

We impose $y \, \tilde{R} \, \underset{\underset{X'}{\uparrow}}{z} \iff$ there is $w \in X$ such that

$$\begin{cases} w R y \\ \sigma(w) R' z \end{cases}$$

on X' it is R'

and $y \tilde{R} y$.

$\mathcal{L} = \{\underset{\underset{\text{binary}}{\uparrow}}{E}\}$ $\quad T = \{$ E is an equiv. relation with infinitely many equiv. classes $\}$

$$\begin{tikzcd}
\mathrm{FinSet}^{op} \arrow[r] & \mathrm{Stone} \\
X \arrow[r, mapsto] & \{\text{linear preorders on } X\} \arrow[d, mapsto] \\
X \arrow[u, "\sigma"] & {}
\end{tikzcd}$$

refl + trans. linear: $x \leq y$ or $y \leq x$.

$$\begin{tikzcd}
R \arrow[d, mapsto] \\
(\sigma \times \sigma)^{-1}[R]
\end{tikzcd}$$

$x_1 R' x_2 \Leftrightarrow \sigma(x_1) R \sigma(x_2)$

$\mathcal{L} = \{R\}$ ← linear

$T = \{$nonempty linear preorders dense without endpoints$\}$

$$\begin{tikzcd}
\mathrm{FinSet}^{op} \arrow[r] & \mathrm{Stone} \\
X \arrow[r, mapsto] & \mathcal{P}(X) \\
Y & \mathcal{P}(Y) \arrow[d, "\sigma^{-1}[-]"] \\
X \arrow[u] & \mathcal{P}(X)
\end{tikzcd}$$

unary

$\mathcal{L} = \{R\}$

$T = \{\exists x R(x), \exists x \neg R(x)\}$

CONCLUSION:

Stone duality

Joyal duality

Type space functor
12
Ult

is the
Stone dual
of the
doctrine of
sentences