Syllabus of

Categorical dualities in logic

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Contents

Introduction

John Baez and James Dolan [Baez and Dolan, 2001] remarked that

> "an equation is only interesting or useful to the extent that the two sides are different."

For instance, compare

$$2 = 2 \qquad \text{with} \qquad \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

The first equality is correct but uninformative: both sides express the same object in the same language. The second one is interesting precisely because it connects two apparently different descriptions, and allows us to switch freely between them depending on which is more convenient for a given computation.

In this course, we will study an analogous phenomenon, but at the level of *mathematical structures*. On one side, we will have algebraic structures that arise naturally in logic — most notably *Boolean algebras*, which provide an algebraic semantics for classical propositional logic. On the other side, we will encounter structures of a completely different nature — in this case, *Stone spaces*, which are certain topological spaces.

The connection between these two worlds is not a literal equality, but a *categorical duality* (a.k.a. categorical dual equivalence): a two-way translation that preserves all information. This means that we can translate a problem about Boolean algebras into a corresponding problem about Stone spaces, and vice versa. In practice, this is useful because on the side of Stone spaces many constructions are simpler and one can use geometric intuition.

A slogan to keep in mind is:

> *Categorical dualities in logic relate "algebras of formulas" to "spaces of models".*

We start with the simplest case: classical propositional logic, modeled by Boolean algebras and Stone spaces (Stone, 1936).

Later, we will see analogous dualities for other logics, such as intuitionistic propositional logic, and possibly touch the first-order setting (where quantifiers enter the picture).

Chapter 1

Classical propositional logic: Stone duality

Key definitions and theorems

In the first part of the course, we will see *Stone duality*: a connection (in the form of a categorical duality) between *Boolean algebras* and *Stone spaces*. Informally, Boolean algebras encode the *syntax* of classical propositional logic (algebras of formulas), while Stone spaces encode its *semantics* (spaces of models).

1.1 From classical propositional logic to Boolean algebras

1.1.1 Syntax: propositional languages and formulas

Definition 1.1 (Propositional language, propositional symbols)**.** A ***(propositional) language*** $\mathcal{L}$ is a set; its elements are called ***propositional symbols*** (or also *propositional variables*).

We use the fancy name "(propositional) language" for a plain set just to declare the usage we want to make out of it.

Propositional symbols are typically denoted by $p, q, r, \dots$.

The connectives of classical propositional logic are

$$\vee, \quad \wedge, \quad \neg, \quad 0, \quad 1,$$

where $\vee$ is a binary operation denoting *or* (sometimes called *join*), $\wedge$ is a binary operation denoting *and* (sometimes called *meet*), $\neg$ is a unary operation denoting *not* (i.e. negation, or complement), 0 is a constant symbol denoting *false* (bottom) and 1 is a constant symbol denoting *true* (top).

Definition 1.2 (Formulas)**.** The set of ***formulas*** $\mathrm{Form}(\mathcal{L})$ is defined inductively as follows:

- every propositional symbol $p \in \mathcal{L}$ is a formula;
- if $\varphi, \psi \in \mathrm{Form}(\mathcal{L})$, then $(\varphi \vee \psi) \in \mathrm{Form}(\mathcal{L})$ and $(\varphi \wedge \psi) \in \mathrm{Form}(\mathcal{L})$;
- if $\varphi \in \mathrm{Form}(\mathcal{L})$, then $(\neg\varphi) \in \mathrm{Form}(\mathcal{L})$;
- 0 and 1 are formulas.

In other words, $\mathrm{Form}(\mathcal{L})$ is the smallest set containing $\mathcal{L}$ and closed under $\vee$, $\wedge$, $\neg$, 0 and 1.

We use the standard abbreviations for implications and bi-implication:

$$\varphi \to \psi \;\coloneqq\; \neg\varphi \vee \psi, \qquad \varphi \leftrightarrow \psi \;\coloneqq\; (\varphi \to \psi) \wedge (\psi \to \varphi).$$

While propositional symbols are usually denoted by $p, q, r, \dots$, formulas are typically denoted by $\varphi, \psi, \sigma, \dots$.

1.1.2 Semantics: valuations and truth tables

Set $2 \coloneqq \{0, 1\}$ (the "set of truth values").

Definition 1.3 (Valuations, interpretation of formulas)**.** A ***valuation*** on $\mathcal{L}$ is a function $v\colon \mathcal{L} \to 2$. Given a valuation v, its unique extension to all formulas is a map

$$\bar{v}\colon \mathrm{Form}(\mathcal{L}) \longrightarrow 2$$

defined by recursion on the complexity of formulas:

$$\bar{v}(p) = v(p) \quad (p \in \mathcal{L}), \qquad \bar{v}(0) = 0, \qquad \bar{v}(1) = 1,$$

$$\bar{v}(\neg\varphi) = \neg\,\bar{v}(\varphi), \qquad \bar{v}(\varphi \vee \psi) = \bar{v}(\varphi) \vee \bar{v}(\psi), \qquad \bar{v}(\varphi \wedge \psi) = \bar{v}(\varphi) \wedge \bar{v}(\psi),$$

where on the right-hand side we use the usual Boolean operations on $2 = \{0, 1\}$, which you can find in Remark 1.4 below.

Remark 1.4 (Boolean operations on $2 = \{0, 1\}$)**.** By convention,

$$0 \wedge 0 = 0, \;\; 0 \wedge 1 = 0, \;\; 1 \wedge 0 = 0, \;\; 1 \wedge 1 = 1,$$

$$0 \vee 0 = 0, \;\; 0 \vee 1 = 1, \;\; 1 \vee 0 = 1, \;\; 1 \vee 1 = 1,$$
$$\neg 0 = 1, \;\; \neg 1 = 0.$$

1.1.3 Semantic equivalence

Definition 1.5 (Semantic equivalence)**.** Let $\varphi, \psi \in \mathrm{Form}(\mathcal{L})$. We write $\varphi \equiv \psi$ and say that φ and ψ are ***semantically equivalent***[a] if
$$\forall v \colon \mathcal{L} \to 2, \qquad \bar{v}(\varphi) = \bar{v}(\psi).$$

[a] In Definition 1.5, "semantic" is in opposition to "syntactic": two formulas are *syntactically equivalent* if they are interprovable in a certain proof system, which we do not have the time to see here. Let me just mention that Stone's Representation Theorem, which will be seen later and which is the core of this chapter, can be seen as an algebraic way to affirm that the syntactic and semantic notions of equivalence coincide.

In other words, φ and ψ are equivalent if and only if they have the same truth table (a function from $\mathcal{L}$ to 2 corresponds to a row of a truth table).

Example 1.6. Let $\mathcal{L} = \{ p, q \}$. Then $p \vee q \equiv q \vee p$, as can be checked via the truth table

p	q	$p \vee q$	$q \vee p$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

Example 1.7. If $\mathcal{L} = \{ p \}$ has one variable, then every formula is equivalent to exactly one of
$$0, \quad p, \quad \neg p, \quad 1.$$
(For instance, $\neg\neg p \equiv p$, $p \vee \neg p \equiv 1$, and $p \wedge \neg p \equiv 0$.) Thus $\mathrm{Form}(\mathcal{L})/{\equiv}$ has 4 elements.

One may prove that, in a language $\mathcal{L} = \{ p, q \}$ with two propositional symbols, there are 16 equivalence classes of formulas.[1]

1.1.4 Adding assumptions: theories and semantic equivalence modulo a theory

We can incorporate semantic assumptions by restricting the class of admissible valuations.

Definition 1.8 (Propositional theory)**.** A ***(propositional) theory*** $\mathcal{T}$ in a propositional language $\mathcal{L}$ is a subset $\mathcal{T} \subseteq \mathrm{Form}(\mathcal{L})$.

Definition 1.9 (Model)**.** Let $\mathcal{T} \subseteq \mathrm{Form}(\mathcal{L})$ be a propositional theory in a propositional language. A ***model*** of $\mathcal{T}$ is a function $v \colon \mathcal{L} \to 2$ such that
$$\forall \sigma \in \mathcal{T}, \qquad \bar{v}(\sigma) = 1.$$
We denote by $\mathrm{Mod}(\mathcal{T})$ the set of models of $\mathcal{T}$.

Example 1.10. Let $\mathcal{L} = \{ p, q \}$ and let $\mathcal{T} = \{ p \vee q \}$. Then $\mathrm{Mod}(\mathcal{T})$ consists of the three valuations
$$(p, q) = (1, 0), \;\; (1, 1), \;\; (0, 1).$$

[1] More generally, if $\mathcal{L}$ is finite of cardinality n, then $|\mathrm{Form}(\mathcal{L})/{\equiv}| = 2^{2^n}$. Indeed, every formula determines a truth function $2^{\mathcal{L}} \to 2$, and vice versa one can prove that every function $2^{\mathcal{L}} \to 2$ is the truth function of a formula (this is called "functional completeness"), so that $|\mathrm{Form}(\mathcal{L})/{\equiv}| = 2^{2^{|\mathcal{L}|}}$. In particular, for $\mathcal{L} = \{ p, q \}$ we have $|\mathrm{Form}(\mathcal{L})/{\equiv}| = 2^{2^2} = 16$.

Definition 1.11 (Semantic equivalence modulo a theory)**.** Let $\mathcal{T} \subseteq \mathrm{Form}(\mathcal{L})$ be a propositional theory and let $\varphi, \psi \in \mathrm{Form}(\mathcal{L})$. We say that φ and ψ are ***semantically equivalent modulo the theory*** $\mathcal{T}$ (or ***relative to*** $\mathcal{T}$, or ***semantically*** $\mathcal{T}$***-equivalent***), and write $\varphi \equiv_{\mathcal{T}} \psi$, if for every $v \in \mathrm{Mod}(\mathcal{T})$ we have

$$\bar{v}(\varphi) = \bar{v}(\psi).$$

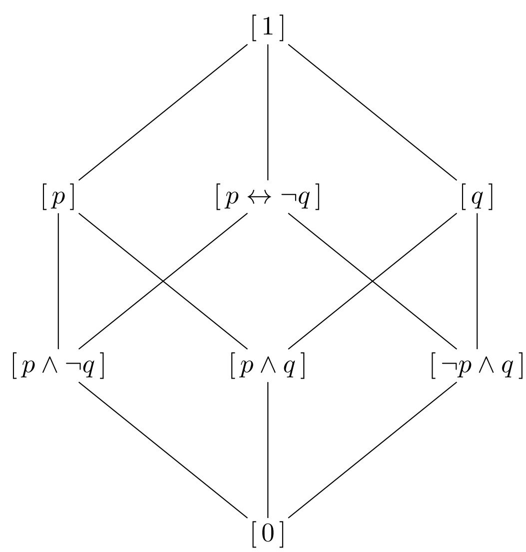
Example 1.12. Let $\mathcal{L} = \{\, p, q \,\}$ and let $\mathcal{T} = \{\, p \vee q \,\}$. One can check that $p \vee q \equiv_{\mathcal{T}} 1$, $\neg p \wedge \neg q \equiv_{\mathcal{T}} 0$ and $p \vee \neg q \equiv_{\mathcal{T}} p$.

The relation $\equiv_{\mathcal{T}}$ is again an equivalence relation, and we write $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ for the corresponding quotient set. It is convenient to partially order equivalence classes by *implication*:

$$\varphi \leq_{\mathcal{T}} \psi \quad :\Longleftrightarrow \quad \forall v \in \mathrm{Mod}(\mathcal{T}),\ \big(\bar{v}(\varphi) = 1 \Rightarrow \bar{v}(\psi) = 1\big).$$

(So "$\varphi \leq_{\mathcal{T}} \psi$" means that φ *implies* ψ on all models of $\mathcal{T}$.)

Example 1.13. Let $\mathcal{L} = \{\, p, q \,\}$ and $\mathcal{T} = \{\, p \vee q \,\}$. Then $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ has 8 elements and its Hasse diagram (with respect to $\leq_{\mathcal{T}}$) can be drawn as follows:



Boolean algebras (whose definition we will see soon) are meant to capture the algebraic structures of the form

$$\langle \mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}; \vee, \wedge, \neg, 0, 1 \rangle,$$

for $\mathcal{L}$ a propositional language and $\mathcal{T}$ a propositional theory in $\mathcal{L}$; here, $\vee, \wedge, \neg, 0, 1$ are defined on $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ by setting

$$[\,\varphi\,] \vee [\,\psi\,] \ \coloneqq\ [\,\varphi \vee \psi\,], \qquad [\,\varphi\,] \wedge [\,\psi\,] \ \coloneqq\ [\,\varphi \wedge \psi\,], \qquad \neg\,[\,\varphi\,] \ \coloneqq\ [\,\neg\varphi\,], \qquad 0 \ \coloneqq\ [\,0\,], \qquad 1 \ \coloneqq\ [\,1\,].$$

(These are well-defined operations.)

In other words, Boolean algebras are meant to capture the algebras of formulas modulo a theory.

1.2 Boolean algebras

The quotient posets in Example 1.13 already display a key phenomenon: the logical connectives $\wedge$ and $\vee$ behave like "infimum" and "supremum" with respect to the implication order.

To recall the definition of infimum and supremum, let P be a poset. For $x, y \in P$, an *infimum* (or *greatest lower bound*) of $\{x, y\}$ is an element $x \wedge y \in P$ such that

$$x \wedge y \leq x, \qquad x \wedge y \leq y,$$

and, for every $z \in P$, if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$. Similarly, a *supremum* (or *least upper bound*) of $\{x, y\}$ is an element $x \vee y \in P$ such that

$$x \leq x \vee y, \qquad y \leq x \vee y,$$

and, for every $z \in P$, if $x \leq z$ and $y \leq z$ then $x \vee y \leq z$.

More generally, for any subset S of a poset P, an ***infimum*** of S (relative to P) is a greatest lower bound of S in P and a ***supremum*** of S (relative to P) is a smallest upper bound of S in P. Infima and suprema, if they exist, are unique. Note that being the infimum of $\varnothing$ relative to P means being the maximum of P, and being the supremum of $\varnothing$ relative to P means being the minimum of P.

One can show that, in the poset $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ ordered by implication, $[\varphi \wedge \psi]$ is the infimum of $[\varphi]$ and $[\psi]$, and $[\varphi \vee \psi]$ is the supremum of $[\varphi]$ and $[\psi]$.[2]

1.2.1 Order-theoretic definition of lattices

Definition 1.14 (Lattice: order theoretic presentation)**.** A ***lattice*** is a poset $(L, \leq)$ in which every pair of elements admits an infimum and a supremum.

Example 1.15 (Power-set lattice)**.** Let X be a set and consider the poset $(\mathscr{P}(X), \subseteq)$. Then $(\mathscr{P}(X), \subseteq)$ is a lattice, with

$$A \wedge B \coloneqq A \cap B \qquad \text{and} \qquad A \vee B \coloneqq A \cup B.$$

1.2.2 Equational definition of lattices

Lattices can also be presented *algebraically*, by taking $\wedge$ and $\vee$ as primitive operations and listing a small family of identities. This is useful because identities are stable under the kind of constructions we will use later (products, subalgebras, quotients).

Definition 1.16 (Lattice: equational presentation)**.** A ***lattice*** is a set L equipped with two binary operations

$$\wedge, \vee \colon L \times L \to L$$

such that:

1. (commutativity) for all $a, b \in L$, $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$;
2. (associativity) for all $a, b, c \in L$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$;
3. (absorption) for all $a, b \in L$, $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

Remark 1.17. If L is a lattice in the equational sense, one can recover an order by setting

$$a \leq b \quad :\Longleftrightarrow \quad a \wedge b = a,$$

equivalently $a \vee b = b$. With this order, the operations $\wedge$ and $\vee$ are precisely infimum and supremum, so Definitions 1.14 and 1.16 are equivalent viewpoints.

[2]Proof: We prove the statement for $\wedge$; the case of $\vee$ is analogous. For every model $v \in \mathrm{Mod}(\mathcal{T})$, if $\bar{v}(\varphi \wedge \psi) = 1$ then $\bar{v}(\varphi) = 1$ and $\bar{v}(\psi) = 1$, hence $[\varphi \wedge \psi] \leq_{\mathcal{T}} [\varphi]$ and $[\varphi \wedge \psi] \leq_{\mathcal{T}} [\psi]$. Now let $[\rho]$ be any lower bound of $[\varphi]$ and $[\psi]$. This means that, for every $v \in \mathrm{Mod}(\mathcal{T})$, $\bar{v}(\rho) = 1$ implies both $\bar{v}(\varphi) = 1$ and $\bar{v}(\psi) = 1$. Therefore $\bar{v}(\rho) = 1$ implies $\bar{v}(\varphi \wedge \psi) = 1$, i.e. $[\rho] \leq_{\mathcal{T}} [\varphi \wedge \psi]$.

1.2.3 Bounded lattices

Definition 1.18 (Bounded lattice)**.** A lattice is ***bounded*** if it has a least element 0 and a greatest element 1, i.e. elements such that $0 \leq a \leq 1$ for all a. Equivalently (in the equational presentation), 0 and 1 satisfy

1. For all a, $a \wedge 0 = 0$ (equivalently: $a \vee 0 = a$),
2. For all a, $a \vee 1 = 1$ (equivalently; $a \wedge 1 = a$).

1.2.4 Distributive lattices

Definition 1.19 (Distributive lattice)**.** A lattice L is ***distributive*** if it satisfies any (and hence both) of the following equivalent[a] conditions:

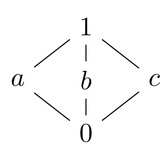
1. for all $a, b, c \in L$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.
2. for all $a, b, c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

[a] For a proof of the equivalence between the two, see [Birkhoff, 1967, Sec. 6, Thm. 9, p. 11].

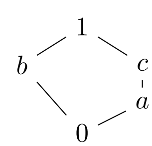
Example 1.20. The two-element chain is distributive:

Example 1.21. For every set X, the power-set lattice $\mathscr{P}(X)$ is distributive.

Two small lattices play a special role as "minimal" obstructions to distributivity. They are usually denoted by M_3 (the *diamond*) and N_5 (the *pentagon*).



M_3



N_5

(1.1)

These are not distributive; one can easily verify that $a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$.

In fact, one can prove that a lattice L is distributive if and only if it does not contain a sublattice isomorphic to M_3 or N_5, i.e., there is no injective map $M_3 \to L$ or $N_5 \to L$ preserving both $\wedge$ and $\vee$.[3]

1.2.5 Boolean algebras

Definition 1.22 (Boolean algebra)**.** A ***Boolean algebra*** is an algebraic structure

$$\langle B; \vee, \wedge, \neg, 0, 1 \rangle$$

such that:

[3] For a textbook reference, see, e.g., [Davey and Priestley, 2002, 4.10].

1. $\langle B; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice;
2. for every $a \in B$,

$$a \wedge \neg a = 0 \qquad \text{and} \qquad a \vee \neg a = 1.$$

The element $\neg a$ is called the *complement* of a.

Remark 1.23. All axioms in the definition of Boolean algebras are *equational*, i.e. they are identities of the form

$$\forall x_1, \ldots, x_n, \quad t_1(x_1, \ldots, x_n) = t_2(x_1, \ldots, x_n)$$

between algebraic terms.

Remark 1.24. A Boolean algebra is completely determined by its underlying partial order. This is because: $\vee, \wedge, 0, 1$ are the binary supremum, binary infimum, smallest element and greatest element, and, in every bounded distributive lattice L, if an element a has a complement (i.e., there is an element b such that $a \wedge b = 0$ and $a \vee b = 1$) it is unique.[4]

Example 1.25. The prototypical example of a Boolean algebra is $2 = \{0, 1\}$ with partial order $0 \leq 1$ is a Boolean algebra. The Boolean operations are those described in Remark 1.4.

Proposition 1.26 ($\mathrm{Form}(\mathcal{L})/\equiv_{\mathcal{T}}$ is a Boolean algebra)**.** *Let $\mathcal{L}$ be a propositional language and $\mathcal{T} \subseteq \mathrm{Form}(\mathcal{L})$ a theory. Then the quotient set $\mathrm{Form}(\mathcal{L})/\equiv_{\mathcal{T}}$ becomes a Boolean algebra by setting*

$$[\varphi] \vee [\psi] \coloneqq [\varphi \vee \psi], \qquad [\varphi] \wedge [\psi] \coloneqq [\varphi \wedge \psi], \qquad \neg[\varphi] \coloneqq [\neg\varphi], \qquad 0 \coloneqq [0], \qquad 1 \coloneqq [1].$$

Proof sketch. Step 1: the operations are well defined. We show this for $\vee$; the cases of $\wedge$ and $\neg$ are analogous. Suppose $\varphi \equiv_{\mathcal{T}} \varphi'$ and $\psi \equiv_{\mathcal{T}} \psi'$. Then for every $v \in \mathrm{Mod}(\mathcal{T})$ we have $\bar{v}(\varphi) = \bar{v}(\varphi')$ and $\bar{v}(\psi) = \bar{v}(\psi')$. Using the truth table for $\vee$ in $2 = \{0, 1\}$ we compute:

$$\bar{v}(\varphi \vee \psi) = \bar{v}(\varphi) \vee \bar{v}(\psi) = \bar{v}(\varphi') \vee \bar{v}(\psi') = \bar{v}(\varphi' \vee \psi').$$

Hence $\varphi \vee \psi \equiv_{\mathcal{T}} \varphi' \vee \psi'$, and so $[\varphi] \vee [\psi]$ does not depend on the choice of representatives. The constants $0 = [0]$ and $1 = [1]$ are trivially well defined.

Step 2: the Boolean algebra identities hold. Example (one Boolean identity: complements). Fix $\varphi \in \mathrm{Form}(\mathcal{L})$.

For every $v \in \mathrm{Mod}(\mathcal{T})$ we compute

$$\bar{v}(\varphi \wedge \neg\varphi) = \bar{v}(\varphi) \wedge \bar{v}(\neg\varphi) = \bar{v}(\varphi) \wedge \neg\bar{v}(\varphi) = 0 = \bar{v}(0),$$

where we used the recursive definition of $\bar{v}$ and the fact that $b \wedge \neg b = 0$ for all $b \in 2$. Hence $\varphi \wedge \neg\varphi \equiv_{\mathcal{T}} 0$, and therefore

$$[\varphi] \wedge \neg[\varphi] = [0] = 0.$$

This is the general mechanism: any identity between Boolean terms can be checked pointwise in 2 under every valuation. To give more details, let $t_1(x_1, \ldots, x_n) = t_2(x_1, \ldots, x_n)$ be any Boolean algebra

[4] This is not true for arbitrary bounded lattice: for example, in the bounded lattice M_3 in (1.1), both b and c are complements of a, and in N_5 both a and c are complements of b.

identity (an equation between terms built from $\vee, \wedge, \neg, 0, 1$). To check it in $\mathrm{Form}(\mathcal{L})/{\equiv_\mathcal{T}}$, pick arbitrary formulas $\varphi_1, \ldots, \varphi_n \in \mathrm{Form}(\mathcal{L})$ and consider the two formulas $t_1(\varphi_1, \ldots, \varphi_n)$ and $t_2(\varphi_1, \ldots, \varphi_n)$. For every $v \in \mathrm{Mod}(\mathcal{T})$, the evaluation map $\bar{v} \colon \mathrm{Form}(\mathcal{L}) \to 2$ respects the connectives, hence

$$\bar{v}\big(t_i(\varphi_1, \ldots, \varphi_n)\big) = t_i\big(\bar{v}(\varphi_1), \ldots, \bar{v}(\varphi_n)\big) \qquad (i = 1, 2).$$

Since the identity $t_1 = t_2$ holds in the two-element Boolean algebra 2, the right-hand sides are equal for all $v \in \mathrm{Mod}(\mathcal{T})$, so $t_1(\varphi_1, \ldots, \varphi_n) \equiv_\mathcal{T} t_2(\varphi_1, \ldots, \varphi_n)$. Therefore, the induced operations on equivalence classes satisfy all Boolean algebra axioms. ■

Definition 1.27 (Lindenbaum–Tarski algebra)**.** Let $\mathcal{T}$ be a propositional theory in a propositional language $\mathcal{L}$. The Boolean algebra

$$\langle \mathrm{Form}(\mathcal{L})/{\equiv_\mathcal{T}}; \vee, \wedge, \neg, 0, 1 \rangle$$

with operations

$$[\,\varphi\,] \vee [\,\psi\,] \coloneqq [\,\varphi \vee \psi\,], \qquad [\,\varphi\,] \wedge [\,\psi\,] \coloneqq [\,\varphi \wedge \psi\,], \qquad \neg\,[\,\varphi\,] \coloneqq [\,\neg\varphi\,], \qquad 0 \coloneqq [\,0\,], \qquad 1 \coloneqq [\,1\,]$$

is called the ***Lindenbaum–Tarski algebra*** of $\mathcal{T}$.

Later on, we will see that also the converse holds: every Boolean algebra is isomorphic to $\mathrm{Form}(\mathcal{L})/{\equiv_\mathcal{T}}$ for some language $\mathcal{L}$ and some theory $\mathcal{T}$. This will be a consequence of Stone's Representation Theorem (or of the crucial lemmas used in proving Stone's Representation Theorem). It will provide a guarantee that the definition of Boolean algebras is "the correct one".

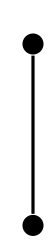
Example 1.28 (Power set Boolean algebras)**.** Let X be a set. The power set $\mathscr{P}(X)$ with the inclusion order is a Boolean algebra, with operations

$$A \wedge B \coloneqq A \cap B, \qquad A \vee B \coloneqq A \cup B, \qquad \neg A \coloneqq X \setminus A, \qquad 0 \coloneqq \varnothing, \qquad 1 \coloneqq X.$$

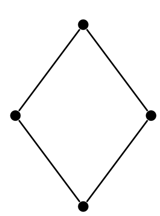
See the footnote for a proof.[5]

Below, we draw the Hasse diagrams of the power sets of sets of cardinality 0, 1, 2, and 3. These are the smallest Boolean algebras, and probably the only ones that one can draw without losing one's sight. (The next one has 16 elements.)

The *trivial Boolean algebra*, of cardinality 1, isomorphic to $\mathscr{P}(\varnothing)$

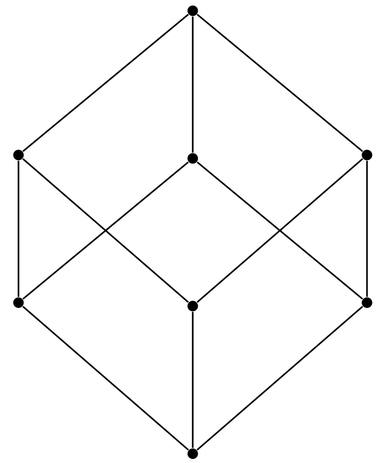


The Boolean algebra $2 = \{0, 1\}$, of cardinality 2, isomorphic to $\mathscr{P}(\{\,x\,\})$



The *diamond*: a Boolean algebra of cardinality 4, isomorphic to $\mathscr{P}(\{\,x, y\,\})$

[5] One way to prove that it is a Boolean algebra is by observing it is isomorphic to the power $\prod_{x \in X} 2$ of 2, that 2 is a Boolean algebra, and that equationally defined classes of algebras are closed under products (by the easy direction in Birkhoff's theorem).



The *cube*: a Boolean algebra of cardinality 8, isomorphic to $\mathscr{P}(\{x, y, z\})$

There are examples of Boolean algebras that are not isomorphic to any power set. The following one is probably the simplest example.

Example 1.29 (The algebra of finite and cofinite subsets of $\mathbb{N}$)**.** Consider the collection

$$\mathrm{FC}(\mathbb{N}) \;:=\; \{\, A \subseteq \mathbb{N} \mid A \text{ is finite, or cofinite} \,\} \;\subseteq\; \mathscr{P}(\mathbb{N})$$

where "*A cofinite*" means that $\mathbb{N} \setminus A$ is finite. To show that this is indeed a Boolean algebra, the following is of help.

Remark 1.30 (Boolean subalgebras of power sets)**.** If $\mathcal{A} \subseteq \mathscr{P}(X)$ contains $\varnothing$ and X and is closed under $\cup$, $\cap$ and complement in X, then $\mathcal{A}$ is a Boolean algebra, as well.

This follows from the fact that Boolean algebras are equationally definable and hence closed under subalgebras.

Now one can prove that $\mathrm{FC}(\mathbb{N})$ is a Boolean algebra:

- $\varnothing$ and $\mathbb{N}$ belong to $\mathrm{FC}(\mathbb{N})$;
- $\mathrm{FC}(\mathbb{N})$ is closed under complement in $\mathbb{N}$: if A is finite then $\mathbb{N} \setminus A$ is cofinite, and if A is cofinite then $\mathbb{N} \setminus A$ is finite;
- $\mathrm{FC}(\mathbb{N})$ is closed under unions: if A and B are finite then $A \cup B$ is finite, while if at least one of A, B is cofinite then $A \cup B$ is cofinite.
- Analogously, $\mathrm{FC}(\mathbb{N})$ is closed under intersections.

The Boolean algebra $\mathrm{FC}(\mathbb{N})$ is infinite but much smaller than $\mathscr{P}(\mathbb{N})$: for instance, the set of even numbers is neither finite nor cofinite, hence it does not belong to $\mathrm{FC}(\mathbb{N})$. Moreover, $\mathrm{FC}(\mathbb{N})$ is *countably infinite* (i.e., in bijection with $\mathbb{N}$). This also shows that it is not isomorphic to any power set, since no power set is countably infinite: if X is finite then $\mathscr{P}(X)$ is finite, and if X is infinite then $\mathscr{P}(X)$ is uncountable.

Remark 1.31 (A different obstruction: completeness)**.** More generally, for any set X, one can consider the Boolean subalgebra

$$\mathrm{FC}(X) := \{\, A \subseteq X \mid A \text{ is finite or cofinite} \,\} \subseteq \mathscr{P}(X).$$

If X is infinite, then $\mathrm{FC}(X)$ is *not complete* as a Boolean algebra: there are families that do not admit a supremum. For instance, choose a subset $Y \subseteq X$ such that both Y and $X \setminus Y$ are infinite, and consider the family $\{\, \{\, y \,\} \mid y \in Y \,\} \subseteq \mathrm{FC}(X)$. Its union in $\mathscr{P}(X)$ is Y, which does not belong to $\mathrm{FC}(X)$. Moreover, there is no least element of $\mathrm{FC}(X)$ containing all singletons $\{\, y \,\}$ (one can always remove a point from any upper bound and still obtain a cofinite upper bound), hence the family has no supremum in $\mathrm{FC}(X)$. By contrast, every power set $\mathscr{P}(X)$ is complete (arbitrary unions exist), so $\mathrm{FC}(X)$ for X infinite cannot be isomorphic to any power set.

1.3 Stone's Representation Theorem

Remark 1.30 gives a zoo of examples: every *Boolean subalgebra* of some power set is a Boolean algebra. Stone's Representation Theorem states that all Boolean algebras are of this form! We state it now, and we will prove it later Theorem 1.51.

Theorem (Stone's Representation Theorem for Boolean algebras)**.** *For every Boolean algebra B there is a set X such that B is isomorphic to a Boolean subalgebra of the power set $\mathscr{P}(X)$.*

This means that for every Boolean algebra B there are a set X and an injective map

$$\iota\colon B \hookrightarrow \mathscr{P}(X)$$

such that, under ι, the operations on B correspond to intersection, union, complement, empty set and whole set in X:

$$\iota(a \wedge b) = \iota(a) \cap \iota(b), \qquad \iota(a \vee b) = \iota(a) \cup \iota(b), \qquad \iota(\neg a) = X \setminus \iota(a), \qquad \iota(0) = \varnothing, \qquad \iota(1) = X.$$

The next goal is to prove Stone's Representation Theorem for Boolean algebras. We will first need to present auxiliary notions and lemmas.

Given a Boolean algebra B, how can we find a set X such that B embeds into the power set $\mathscr{P}(X)$ of X, as required by the statement of Stone's Representation Theorem?

Idea, from a logical perspective: In the special Boolean algebra $\mathrm{Form}(\mathcal{L})/{\equiv_\mathcal{T}}$ (for $\mathcal{L}$ a language and $\mathcal{T}$ a theory), an equivalence class $[\varphi]$ can be identified with the set of models $v \in \mathrm{Mod}(\mathcal{T})$ such that $\bar{v}(\varphi) = 1$; i.e., a formula can be identified with the models that satisfy it. This gives a very concrete embedding into a power set: the power set of models of $\mathcal{T}$. So, the idea for $B = \mathrm{Form}(\mathcal{L})/{\equiv_\mathcal{T}}$ is to take

$$X = \mathrm{Mod}(\mathcal{T}).$$

To translate this idea to a general Boolean algebra B, we note that a model

$$v\colon \mathcal{L} \longrightarrow 2$$

of $\mathcal{T}$ induces a function

$$\bar{v}\colon \mathrm{Form}(\mathcal{L}) \longrightarrow 2$$

which passes to the quotient

$$\begin{aligned} \mathrm{Form}(\mathcal{L})/{\equiv_\mathcal{T}} &\longrightarrow 2 \\ [\varphi] &\longmapsto \bar{v}(\varphi). \end{aligned}$$

In fact, one can prove that the models of $\mathcal{T}$ are in bijection with the Boolean homomorphisms (i.e., functions preserving all Boolean connectives, see Definition 1.32 below) from $\mathrm{Form}(\mathcal{L})/{\equiv_\mathcal{T}}$ to 2. This suggests that, for a general Boolean algebra B, we shall take

$$X = \hom(B, 2),$$

the set of homomorphisms from B to 2.

Idea, from another perspective: If $B \subseteq \mathscr{P}(X)$ is a Boolean subalgebra of $\mathscr{P}(X)$, then every element $x \in X$ induces a function

$$\begin{aligned} B &\longrightarrow 2 \\ A &\longmapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

which is a Boolean homomorphism. This suggests a close relationship between X and $\hom(B, 2)$. This suggests, from yet another perspective, that to prove Stone's Representation Theorem we shall take

$$X = \hom(B, 2).$$

Definition 1.32 (Boolean homomorphism)**.** Let A and B be Boolean algebras. A ***Boolean homomorphism*** (or, simply, a *homomorphism*) $f\colon A \to B$ is a function such that, for all $a, b \in A$,

$$f(1) = 1, \qquad f(a \wedge b) = f(a) \wedge f(b), \qquad f(\neg a) = \neg f(a).$$

Remark 1.33. From the defining equations one immediately gets $f(0) = 0$ and $f(a \vee b) = f(a) \vee f(b)$. In other words, a Boolean homomorphism is a function that preserves all the basic logical operations: $\wedge$, $\vee$, $\neg$, 0, 1.

1.3.1 Ultrafilters

A homomorphism $f\colon B \to 2$ can be encoded via $f^{-1}[\{ 1 \}]$. The subsets of B arising as $f^{-1}[\{ 1 \}]$ for some homomorphism $f\colon B \to 2$ can be characterized as the *ultrafilters*. To define this, we first define filters (of which ultrafilters are special instances) and ideals.

Definition 1.34 (Filter)**.** A ***filter*** of a Boolean algebra B is a subset $F \subseteq B$ such that:

1. F is *upward closed*, i.e., if $x \in F$ and $x \leq y$, then $y \in F$;
2. F is *closed under finite meets*, i.e.,
   (a) if $x, y \in F$, then $x \wedge y \in F$;
   (b) $1 \in F$.

A filter F is *proper* if $0 \notin F$.

Turning the definition of a filter upside down, we get the following.

Definition 1.35 (Ideal)**.** An ***ideal*** of a Boolean algebra B is a subset $I \subseteq B$ such that:

1. I is *downward closed*, i.e., if $x \in I$ and $y \leq x$, then $y \in I$;
2. I is *closed under finite joins*, i.e.,
   (a) if $x, y \in I$, then $x \vee y \in I$;
   (b) $0 \in I$.

Definition 1.36 (Ultrafilter)**.** An ***ultrafilter*** of a Boolean algebra is a filter whose complement is an ideal.

Spelling out the details, this means that an ultrafilter is a filter F such that

1. if $x \vee y \in F$, then $x \in F$ or $y \in F$;
2. $0 \notin F$.

There is also another useful characterization of ultrafilters.

Lemma 1.37 (Ultrafilters via negation)**.** *The ultrafilters of a Boolean algebra B are precisely the filters F such that, for every $x \in B$, exactly one between x and $\neg x$ belongs to F.*

Proof. ($\Rightarrow$). Suppose that F is an ultrafilter, and let $x \in B$. Since $1 = x \vee \neg x \in F$, we have $x \in F$ or $\neg x \in F$. Moreover, they cannot both lie in F because $x \wedge \neg x = 0 \notin F$.
($\Leftarrow$). Suppose that F is a filter such that, for every $x \in B$, exactly one between x and $\neg x$ belongs to F.

- Suppose $x \vee y \in F$, and let us prove that $x \in F$ or $y \in F$. If $x \in F$, we are done. Otherwise, from $x \notin F$ and from the hypothesis on F, we deduce $\neg x \in F$. Then, using distributivity,

$$F \ni (x \vee y) \wedge \neg x = (x \wedge \neg x) \vee (y \wedge \neg x) = y \wedge \neg x.$$

, By upward closure this implies $y \in F$ (since $y \wedge \neg x \leq y$). $x \in F$ or $y \in F$.

- Since 1 belongs to F, its negation $\neg 1 = 0$ does not. ∎

Proposition 1.38 (Ultrafilters encode homomorphisms to 2)**.** *For every Boolean algebra B, there is a bijection*

$$\begin{aligned} \hom(B, 2) &\longleftrightarrow \{\, \textit{ultrafilters of } B \,\}, \\ h &\longmapsto h^{-1}[\{\,1\,\}], \\ \left(\chi_U : x \mapsto \begin{cases} 1, & x \in U, \\ 0, & x \notin U, \end{cases}\right) &\longleftarrow U. \end{aligned}$$

Proof sketch. If $h \colon B \to 2$ is a homomorphism, set $U \coloneqq h^{-1}[\{\,1\,\}]$. Then $1 \in U$, U is upward closed and closed under meets, so it is a filter; it is proper since $h(0) = 0$. Moreover, for each $x \in B$, $h(\neg x) = \neg h(x)$, and hence exactly one of x and $\neg x$ lands in 1. Thus, U satisfies the conditions in Lemma 1.37 and so is an ultrafilter.
Conversely, if U is an ultrafilter, define $h_U \colon B \to 2$ by $h_U(x) = 1$ if and only if $x \in U$. The ultrafilter axioms ensure that h_U preserves $\wedge$, $\neg$, and 1, hence h_U is a homomorphism. ∎

Therefore, ultrafilters of B are encodings of homomorphisms from B to 2. Thus, we will use the set of ultrafilters of B as the set X such that B embeds into $\mathscr{P}(X)$.

Definition 1.39 (Principal filter)**.** The ***principal filter generated by an element*** a of a Boolean algebra B is

$$\uparrow a \;\coloneqq\; \{\, x \in B \mid a \leq x \,\}.$$

Therefore, a filter (and, in particular, an ultrafilter) is called ***principal*** if it has a minimum.

Remark 1.40. If B is finite, then every filter is principal.

We will soon see that $\uparrow a$ is an ultrafilter if and only if a satisfies the following simple condition.

Definition 1.41 (Atom)**.** An ***atom*** of a Boolean algebra B is a minimal element of $B \setminus \{\,0\,\}$.

Proposition 1.42 (Atoms and ultrafilters)**.**

1. *For every Boolean algebra B we have a bijection*

$$\begin{aligned} \{\, \textit{atoms of } B \,\} &\longleftrightarrow \{\, \textit{principal ultrafilters of } B \,\} \qquad (*) \\ a &\longmapsto \uparrow a \\ \min U &\longleftarrow U. \end{aligned}$$

2. *If B is finite, every ultrafilter is principal and so $*$ gives a bijection between atoms and ultrafilters.*

Proof. We only need to prove that the two functions are well defined; that they are mutually inverse will then be immediate.

To prove that the left-to-right function is well defined, let us assume that a is an atom, and let us prove that $U := \uparrow a$ is an ultrafilter. It is a filter (easy). $0 \notin U$ because $a > 0$. Suppose $x \vee y \in \uparrow a$, i.e., $x \vee y \geq a$. Then, by distributivity,

$$a = a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y).$$

Therefore, since the only elements below a are 0 and a, and the join of $(a \wedge x)$ and $(a \wedge y)$ is a, at least one of the two is a (otherwise they would both be 0 and so their join would be 0). If $a \wedge x = a$ then $x \in \uparrow a$, and if $a \wedge y = a$ then $y \in \uparrow a$. Thus, U is an ultrafilter.

To prove that the right-to-left function is well defined, let us assume that $\uparrow a$ is a principal ultrafilter, and let us prove that $\min U$ is an atom. We have $a \neq 0$ (otherwise 0 would belong to U). If $0 < x < a$, then $x \notin \uparrow a$, and hence $\neg x \in \uparrow a$ by Lemma 1.37. Thus $a \leq \neg x$, and since $x \leq a$ we get $x \leq \neg x$, hence $x = x \wedge \neg x = 0$, a contradiction. Therefore, there is no x with $0 < x < a$, so a is an atom. ■

Example 1.43. The trivial Boolean algebra of cardinality 1 has no ultrafilters (and no homomorphisms to 2). The Boolean algebra $2 = \{0, 1\}$ has exactly one ultrafilter, namely $\{1\}$. The "diamond" Boolean algebra of cardinality 4 has exactly two ultrafilters, corresponding to its two atoms. The "cube" Boolean algebra of cardinality 8 has exactly three ultrafilters, corresponding to its three atoms.

In an infinite Boolean algebra, an ultrafilter may fail to be principal, i.e., to have a minimum. The following gives an example.

Example 1.44 (Ultrafilters of FC(ℕ))**.** For each $n \in \mathbb{N}$, the set

$$U_n \ := \ \{ A \in \mathrm{FC}(\mathbb{N}) \mid n \in A \}$$

is a (principal) ultrafilter on the finite-cofinite algebra FC(ℕ) on ℕ.

In addition, the family of cofinite sets

$$U_\infty \ := \ \{ A \in \mathrm{FC}(\mathbb{N}) \mid A \text{ is cofinite} \}$$

is an ultrafilter on FC(ℕ); it is not principal, since it has no minimum. As a simple exercise, you can prove that it has no other ultrafilters, i.e., that

$$\mathrm{Ult}(\mathrm{FC}(\mathbb{N})) = \{ U_n \mid n \in \mathbb{N} \} \cup \{ U_\infty \}.$$

A proof is in the footnote.[6]

Remark 1.45 (Ultrafilters on a power set)**.** What are the ultrafilters of $\mathscr{P}(X)$, for X a set? First of all, for every $x \in X$, we have an ultrafilter

$$U_x := \{ A \in \mathscr{P}(X) \mid x \in A \}.$$

These are precisely the *principal* ultrafilters.

- If X is finite, these are the only ultrafilters (see Proposition 1.42).
- For X infinite, does $\mathscr{P}(X)$ have nonprincipal ultrafilters? This is a nontrivial question; we will see that, if we assume the axiom of choice, we can prove the existence of nonprincipal ultrafilters; this will follow from the Boolean Prime Ideal Theorem (Theorem 1.49) proved below. Do not hope to find an explicit description of any of them, though.

[6] Let U be an ultrafilter on FC(ℕ). If U contains a finite set A, write A as a finite union of singletons: $A = \{n_1\} \cup \cdots \cup \{n_k\}$. Since U is an ultrafilter, the primeness condition in Definition 1.36 implies that some singleton $\{n_i\}$ belongs to U. Then U must coincide with U_{n_i}: if $B \in \mathrm{FC}(\mathbb{N})$ contains n_i, then $\{n_i\} \subseteq B$, hence $B \in U$ by upward closure; if $n_i \notin B$, then $n_i \in \mathbb{N} \setminus B$ and $\mathbb{N} \setminus B \in U$, hence $B \notin U$. If instead U contains no finite set, then for every finite $F \subseteq \mathbb{N}$ we have $F \notin U$, hence $\mathbb{N} \setminus F \in U$ by Lemma 1.37. Thus U contains every cofinite set, i.e. $U_\infty \subseteq U$. Since U_∞ is itself an ultrafilter, maximality forces $U = U_\infty$.

1.3.2 Stone's Representation Theorem

Let us now take $X = \mathrm{Ult}(B)$ and define the embedding $B \hookrightarrow \mathscr{P}(X)$ required by Stone's Representation Theorem.

Definition 1.46 (Stone map)**.** For a Boolean algebra B, we define the ***Stone map***

$$\begin{aligned} \eta_B \colon B &\longrightarrow \mathscr{P}(\mathrm{Ult}(B)) \\ b &\longmapsto \{\, U \in \mathrm{Ult}(B) \mid b \in U \,\}. \end{aligned}$$

Logical intuition: the set $\eta_B(b)$ can be thought of as the set of "models" in which the "formula" b holds. To prove Stone's Representation Theorem it suffices to show that ...

1. η_B is a Boolean homomorphism (easy),
2. η_B is injective (hard).

Lemma 1.47 (The Stone map is a homomorphism)**.** *The Stone map η_B is a Boolean homomorphism, i.e. for all $a, b \in B$,*

$$\eta_B(1) = \mathrm{Ult}(B), \qquad \eta_B(a \wedge b) = \eta_B(a) \cap \eta_B(b), \qquad \eta_B(\neg a) = \mathrm{Ult}(B) \setminus \eta_B(a),$$

(and hence also $\eta_B(a \vee b) = \eta_B(a) \cup \eta_B(b)$ and $\eta_B(0) = \varnothing$).

Proof. The equality $\eta_B(1) = \mathrm{Ult}(B)$ is immediate: every ultrafilter contains 1.
For the binary meet, let $U \in \mathrm{Ult}(B)$. Then

$$U \in \eta_B(a \wedge b) \iff a \wedge b \in U \iff (a \in U \text{ and } b \in U) \iff U \in \eta_B(a) \cap \eta_B(b),$$

where the middle equivalence uses upward closure and closure under binary meets of U.
For complements, using Lemma 1.37 we have

$$U \in \eta_B(\neg a) \iff \neg a \in U \iff a \notin U \iff U \notin \eta_B(a).$$ ■

Remark 1.48 ("Hard part")**.** To conclude the proof of Theorem 1.51, it remains to show that η_B is injective. This is a separation statement: given two distinct elements of B, we must find an ultrafilter containing one but not the other. The required separation will follow from the Boolean Prime Ideal Theorem, proved next.

Let us recall Zorn's Lemma:

Let $(P, \leq)$ be a partially ordered set. If

1. $P \neq \varnothing$,
2. and every nonempty chain[7] in P has an upper bound in P,

then P has a maximal element.

Recall that Zorn's lemma is not provable in ZF, and is equivalent to the Axiom of Choice over ZF. We will use it in the following, which in turn we will use to conclude the proof of Stone's Representation Theorem.

Theorem 1.49 (Boolean Prime Ideal Theorem)**.** *Let B be a Boolean algebra, let F be a filter, and let I be an ideal such that $F \cap I = \varnothing$. Then there is an ultrafilter U on B such that $F \subseteq U$ and $U \cap I = \varnothing$.*

[7] A chain is a totally ordered poset. Thus, a *chain in P* is a subset S of P such that, for all $x, y \in S$, either $x \leq y$ or $y \leq x$.

Proof. Consider the poset

$$\mathcal{P} \;:=\; \{\, (G,J) \mid G \text{ is a filter},\ J \text{ is an ideal},\ F \subseteq G,\ I \subseteq J,\ G \cap J = \varnothing \,\},$$

ordered by componentwise inclusion: $(G,J) \leq (G',J')$ iff $G \subseteq G'$ and $J \subseteq J'$.

Nonemptiness. This poset is nonempty since $(F,I) \in \mathcal{P}$.

Nonempty chains have upper bounds. Let $\mathcal{C} \subseteq \mathcal{P}$ be a nonempty chain and set

$$G^* := \bigcup_{(G,J)\in\mathcal{C}} G, \qquad J^* := \bigcup_{(G,J)\in\mathcal{C}} J.$$

It is straightforward to prove that $(G^*, J*)$ belongs to $\mathcal{P}$: one checks that G^* is a filter, J^* is an ideal, $F \subseteq G^*$, $I \subseteq J^*$ and $G^* \cap J^* = \varnothing$. See the footnote for details. Thus $(G^*, J^*) \in \mathcal{P}$ is an upper bound of $\mathcal{C}$.

By Zorn's lemma, $\mathcal{P}$ has a maximal element (U,K). At this point, the key step is to show that the maximal pair (U,K) "splits" B: once we know that $U \cup K = B$, the disjointness $U \cap K = \varnothing$ forces $K = B \setminus U$. Since K is an ideal, this will imply that $B \setminus U$ is an ideal, and hence U is an ultrafilter.

We claim that $U \cup K = B$. Suppose not, and pick $x \in B \setminus (U \cup K)$.

Let U' be the filter generated by $U \cup \{\, x \,\}$. Concretely,

$$U' = \{\, y \in B \mid \exists u \in U,\ (u \wedge x) \leq y \,\}.$$

If $U' \cap K = \varnothing$, then $(U', K) \in \mathcal{P}$ strictly extends (U,K), contradicting maximality. Hence there are $u \in U$ and $k \in K$ such that $u \wedge x \leq k$, and thus $u \wedge x \in K$.

Similarly, let K' be the ideal generated by $K \cup \{\, x \,\}$. Concretely,

$$K' = \{\, y \in B \mid \exists k \in K,\ y \leq (k \vee x) \,\}.$$

If $U \cap K' = \varnothing$, then $(U, K') \in \mathcal{P}$ strictly extends (U,K), again contradicting maximality. Hence there are $u' \in U$ and $k' \in K$ such that $u' \leq k' \vee x$.

Set $w := u \wedge u' \in U$. Then $w \wedge x \leq u \wedge x \in K$, hence $w \wedge x \in K$ (since K is downward closed), and also $w \leq u' \leq k' \vee x$. Using distributivity,

$$w = w \wedge (k' \vee x) = (w \wedge k') \vee (w \wedge x).$$

Now $w \wedge k' \leq k'$, hence $w \wedge k' \in K$, and we already know $w \wedge x \in K$. Since K is closed under finite joins, we conclude $w \in K$. This contradicts $w \in U$ and $U \cap K = \varnothing$. Therefore $U \cup K = B$. This proves that U is an ultrafilter. It is clear that $F \subseteq U$ and $U \cap I = \varnothing$. ∎

Corollary 1.50 (Ultrafilters separate elements)**.** *If $a \not\leq b$ in B, then there is an ultrafilter $U \in \mathrm{Ult}(B)$ such that $a \in U$ and $b \notin U$.*

Proof. Let $F := \uparrow\! a$ be the principal filter generated by a, and let

$$I := \downarrow b := \{\, x \in B \mid x \leq b \,\}$$

be the principal ideal generated by b. We notice that $F \cap I = \varnothing$, since otherwise there would be $x \in \uparrow\! a \cap \downarrow\! b$, which would imply $a \leq x \leq b$, which would contradict $a \not\leq b$. By Theorem 1.49 there is an ultrafilter U such that $F \subseteq U$ and $U \cap I = \varnothing$. In particular, $a \in U$ and $b \notin U$. ∎

We are now ready to prove:

Theorem 1.51 (Stone's Representation Theorem for Boolean algebras [Stone, 1936])**.** *For every Boolean algebra B there is a set X such that B is isomorphic to a Boolean subalgebra of the power set $\mathscr{P}(X)$.*

Proof. In Lemma 1.47 we proved that the Stone map $\eta_B : B \to \mathscr{P}(\mathrm{Ult}(B))$ defined in Definition 1.46 is a Boolean homomorphism.
We are left to prove it is injective. Let $a \neq b$ in B. Then either $a \not\leq b$ or $b \not\leq a$. Assume $a \not\leq b$. (The proof in the other case is perfectly symmetrical.) By Corollary 1.50, there is $U \in \mathrm{Ult}(B)$ with $a \in U$ and $b \notin U$. Thus $U \in \eta_B(a)$ but $U \notin \eta_B(b)$, so $\eta_B(a) \neq \eta_B(b)$. This shows injectivity. ■

Remark 1.52 (Choice)**.** To prove Stone's Representation Theorem we used the axiom of choice (since we used Zorn's lemma to prove the Boolean Prime Ideal Theorem). In fact, Stone's Representation Theorem is not provable in ZF alone. Over ZF, Stone's Representation Theorem for Boolean algebras is equivalent to the Boolean Prime Ideal Theorem; see, for instance, [Jech, 1973, Sec. 2.6]. The Boolean Prime Ideal Theorem is weaker than the Axiom of Choice, in the sense that it follows from Choice (via Zorn's lemma), but it does not imply Choice [Halpern and Lévy, 1971].

Remark 1.53 (Non-uniqueness and a preview of Stone spaces)**.** A representation of a Boolean algebra as a subalgebra of a power set is generally *not unique.* For instance, the two-element Boolean algebra 2 embeds into $\mathscr{P}(\{x\})$, but it also embeds into $\mathscr{P}(\{x, y\})$ in many different ways. The construction via $\eta_B : B \hookrightarrow \mathscr{P}(\mathrm{Ult}(B))$ is canonical in the sense that it uses the intrinsic set $\mathrm{Ult}(B)$ of ultrafilters.

The canonical representation also satisfies an additional "compactness-type" property: roughly speaking, if the family $\{\eta_B(b) \mid b \in B\}$ covers $\mathrm{Ult}(B)$, then already finitely many $\eta_B(b)$'s cover $\mathrm{Ult}(B)$. This will become literal compactness once we put a topology on $\mathrm{Ult}(B)$.

Indeed, in the next lecture we will equip $\mathrm{Ult}(B)$ with a natural topology generated by the basic sets $\eta_B(b)$, obtaining the *Stone space* associated to B.

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